Transition in Pipe Flow

Nonlinear Mechanisms

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Abstract

Pipe flow is perhaps the classic problem of fluid dynamics. Its simplicity of form lends itself admirably to experimenters but conceals a wealth of unanswered questions. Despite Reynolds’ conducting his seminal experiments over a century ago, few formal results are known for this flow. Over the years many simplifications have been considered, including linearisation, azimuthal invariance and streamwise independence. It is becoming increasingly accepted that these reductions are an over simplification and the full problem must be considered.

In this thesis we begin by looking for exact solutions of the Navier-Stokes equations in pipe flow. These solutions take advantage of the nonlinearity inherent in the system to sustain themselves against viscous decay. The simplest solutions take the form of travelling waves, steady states which propagate down the pipe at a constant velocity.

The travelling waves explored here appear to be of particular importance in two different senses. One particular travelling wave dominates the dynamics on the boundary that separates flows which will relaminarise from those which will lead to a turbulent episode. This travelling wave is central to the transition scenario. We also explore two new families of travelling waves which are particularly elegant form and appear to organise phase space to the extent that all previously known travelling waves appear to be subsidiaries of them. We also explore the first known relative periodic orbit in pipe flow. This represents the next rung up the ladder in terms of complexity of the flow being calculated.

In the second half of this thesis we consider the transient growth mechanism. We recognise the limitations of a linear study and attempt to circumvent this by including nonlinearity. This allows for significantly enhanced growth due to the nonlinear interaction of scales. It also provides us with new and powerful techniques opening the door to wider problems.
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Both of my parents have played an earlier role. My dad planted and nourished an interest in applied mathematics, and my mum introduced me to the world of scientific computing.

Finally, Zoe - girlfriend, fiancée and imminent wife. She has gone through all of this with me, and has made more sacrifices along the way than I dare to mention.
Author’s Declaration

I declare that the work in this dissertation was carried out in accordance with the Regulations of the University of Bristol. The work is original except where indicated by special reference in the text. No part of the dissertation has been submitted for any other academic award. All views expressed in the dissertation are those of the Author.

SIGNATURE

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Chapter 1

Introduction

There are many well known names associated with experimental investigations into the nature of fluid flowing through a circular pipe. But one name that has eluded fame is that of Edme Mariotte. Mariotte was a well respected French physicist and priest best know for a series of four papers, his *Essais de physique*, published in the late 1670’s. In these he perhaps most notably independently discovered Boyle’s law. Although Boyle had beaten him to it by about fifteen years, Mariotte’s methodology was less inductive and probably more insightful to the underlying physics. Unlike Boyle he noted, for instance, that the temperature must be held constant for the law to hold.

But Mariotte’s work was more diverse than just this as he also discovered that the eye has a blind spot, the pressure of tree sap is greater at the bottom of the tree than at the top and in 1686 (two years after his death) he became the first published author on pipe flow.

Mariotte’s work went largely unregarded and the subject of pipe flow was left alone for the next 150 years. In truth his work had been rather superficial, and so it was not until 1839 that Hagen performed the first exacting set of experiments into the problem. Hagen worked with pipes of diameter 2.5–6 mm and noted that two types of flow could be observed
- one unsteady and complicated, the other unidirectional and smooth. It was this latter regime that Poiseuille independently investigated the following year in 1840. By working with much narrower capillary tubes (with radii varying over the range 0.015 – 0.6 mm) Poiseuille unwittingly pigeon-holed himself into only being able to observe the laminar state. This laminar state now bears their names as Hagen-Poiseuille flow.

The most famous experiment of this flow, and perhaps any flow, was performed by Osbourne Reynolds in 1883. Reynolds took rather more elaborate care over the protocol, and from it he deduced several key results. He observed that while the form of flow created appeared to rely on several independent variables (the speed of the fluid, the width of the pipe and the viscosity of the fluid), these different parameters could be reduced into one overall controlling factor, a function of the mean streamwise velocity ($U$), the pipe diameter ($D$) and the viscosity ($\nu$). This is the celebrated Reynolds number, $Re = UD/\nu$. Having reduced the problem down to one controlling parameter he then noted that for $Re$ less than $\simeq 2000$ he only ever saw the laminar ‘direct’ state. However, if the fluid was allowed to move more rapidly then a more chaotic, ‘sinuous’ flow could develop. The onset of this sinuous flow was by no means assured, and by taking sufficient care over his set up to ensure as little disturbance as possible, he was able to observe the laminar state up to $Re \simeq 13000$.

That the onset of turbulence, as we would now call it, was dependent on sufficient perturbation is the experimental realisation of the mathematical statement that pipe flow is linearly stable. After the experiments of Reynolds many people tried to add theoretical explanation. Lord Rayleigh (1892) proved that Hagen-Poiseuille flow is linearly stable in the inviscid limit, while Sexl (1927) and Herron (1991) demonstrated that it is linearly stable to axisymmetric perturbations for all $Re$. Proof that Hagen-Poiseuille flow is truly linearly stable for all $Re$ has never been formally shown, however it has been shown experimentally (Pfen-
niger, 1961) up to $Re = 10^5$ and numerically (Meseguer and Trefethen, 2003) up to $Re = 10^7$.

These large Reynolds number results are complemented by the demonstration by Joseph and Carmi (1969) that any perturbation to the laminar state must necessarily (exponentially) decay below a energy stability limit of $Re_E = 81.49$.

With the establishment of the linear stability of pipe flow, the question for experimentalists became: what is the form of pipe turbulence? The paper which has probably influenced thinking the most is Wygnanski and Champagne (1973). Here they introduced and delineated two terms which are now ubiquitous in the discussion of pipe turbulence: ‘puffs’ and ‘slugs’. Wygnanski and Champagne suggested that at lower Reynolds number turbulence took the form of a puff, which was investigated further in Wygnanski et al. (1975). This is a localised patch of turbulence, typically around twenty pipe diameters in length. The front end of the puff is wispy and hesitantly defined. The trailing edge is much more clearly defined, at least on the centreline of the pipe, but still indistinct near the wall. This can be seen in figure 1.1(upper). The term ‘equilibrium puff’ was coined by Wygnanski et al., as on a macroscopic scale the puff’s size and shape remain consistent and it propagates down the pipe at a near constant speed. The puff’s progress down the pipe is slower than the average laminar flow. This implies that fluid elements enter the puff from the rear, make their way through the turbulent zone and then emerge out the front individually relaminarising.

The slug appears at higher Reynolds numbers. Both leading and trailing edges are much more clearly defined than for the puff and it expands as it advects along the pipe seemingly indefinitely, or at least until it is of ‘the same magnitude as the pipe length’. It can be seen in figure 1.1(lower).

Figure 1.2 is taken from Wygnanski and Champagne and shows how they believed slugs and puffs are distinctly separated. They do note a
patch of uncertainty. More recently it has been suggested that in fact no clear divide exists. Instead the view is that at lower $Re$, one gets equilibrium puffs (as originally suggested), but as $Re$ is increased the puffs begin to expand and transform. As $Re$ is raised further, the indeterminancy is ended as slugs begin to appear. This is a view supported by Darbyshire and Mullin (1995).

As more and more exacting experiments were attempted, concerns were raised over whether the experimental setup could influence the results. Originally most experiments had been done in a flow driven by a constant pressure gradient, and disturbances were created at the inlet. This second point meant that it was not the developed Hagen-Poiseuille profile which was being disturbed, but some alternative state instead. Rubin et al. (1980) compared results from experiments where the disturbance was at the inlet to experiments where the fully developed flow was subjected to a disturbance localised both in space and time. Happily they found that, for sufficiently large disturbances, the transitional flow structure was the same. The first group to experiment on flow driven by a constant mass-flux was Darbyshire and Mullin. They also found no significant variation between the two driving forces.

Most recently two questions have dominated the pipe flow debate: what amplitude of perturbation is required to trigger turbulence, and what magnitude of $Re$ is required to sustain turbulence. The first of these is actually hinted at in figure 1.2 taken from Wygnanski and Cham-
Figure 1.2: A plot taken from Wygnanski and Champagne (1973) showing how the disturbance amplitude required to trigger turbulence depends on Reynolds number and produces either puffs or slugs depending on the parameters.
Here we have disturbance level required to trigger transition plotted against Reynolds number. We see that increasing $Re$ leads to a decrease in amplitude. Darbyshire and Mullin also noted that the amplitude required decreased with $Re$, but the first to suggest a scaling for this was Hof et al. (2003). In this paper they applied a fully localised disturbance to a mass-flux driven pipe in the form of 6 azimuthally arranged jets. Their results suggested that the necessary amplitude scaled as $A \sim Re^{-\gamma}$ with $\gamma = 1$ over a range of Reynolds numbers from 2000 to 20000. Mellibovsky and Meseguer (2007) managed to reproduce this result numerically - a task which requires a clever way of approximating the perturbation.

Peixinho and Mullin (2007) considered just one jet as a perturbation. When the jet was not angled in the streamwise direction at all they recovered the same scaling as before. However, by varying the angle of the jet they found that $\gamma$ could vary from 1.3 – 1.5.

Mellibovsky and Meseguer (2006) performed numerical investigations of pipe flow where they imposed pairs of streamwise independent rolls as an initial condition. For three pairs of rolls they found $\gamma = 1$, for two pairs $\gamma = 1.1$ and for one pair $\gamma \in [1.35, 1.5]$.

In his original paper, Reynolds noted that he only observed turbulence after an $Re \simeq 2000$. Later papers have suggested a minimal Reynolds number of 1900 (Binnie and Fowler, 1947), 2000 (Wygnanski and Champagne, 1973) and 1760 (Darbyshire and Mullin, 1995). Faisst and Eckhardt (2004) attempted to address this question numerically. For varying Reynolds numbers they ran simulations and noted how long the disturbance took to decay. From this they were able to calculate

$$S(t; Re) = P(\text{the flow is turbulent at time } T > t \text{ for a given } Re).$$

(1.0.1)

For a fixed Reynolds number they found that $S(t)$ decayed exponentially
- indicative of a memoryless process - and that it could be expressed as

\[ S(t; Re) = \exp\{t/\tau\}, \quad \tau(Re) = \frac{1}{(Re_c - Re)} \]  

(1.0.2)

\(Re_c\) denotes a critical Reynolds number. Below this turbulence will always decay given sufficient time, while above it turbulence is permanent. They extrapolated a value of \(Re_c = 2250\).

The work by Faisst and Eckhardt was hampered by numerical limitations. They considered a computational domain which was periodic over a length of \(5D\) in the streamwise direction. In reality, as noted earlier, turbulence in a pipe is localised (at this Reynolds number) and it is questionable how realistic an approximation can be made in this short a domain.

Peixinho and Mullin (2006) addressed the same question, but this time experimentally. Of concern to the authors was the influence of the initial condition. To avoid their results being affected, they began each experimental run at \(Re = 1900\) - larger than where they made their lifetime measurements. At this \(Re\) they perturbed the flow and then allowed it to evolve into an equilibrium puff. Once this apparently stable state had been achieved, the Reynolds number within the pipe was reduced smoothly but rapidly to the desired level and measurements were made. This recovered the same scaling law as Faisst and Eckhardt, but for a lower critical Reynolds number of \(Re_c = 1750 \pm 10\). Due to the length of their pipe they were unable to perform experiments for \(Re > 1740\).

In the same year Hof et al. (2006) performed a similar experiment. While again they agreed that \(S(t)\) took an exponential form, they suggested instead that

\[ \tau(Re) = \exp\{0.0344Re - 59.28\} \]  

(1.0.3)

This implies that there is no critical Reynolds number and instead tur-
bulence remains transient for all $Re$.

In order to try and reconcile the differences of these two experiments, Willis and Kerswell (2007) undertook the same problem numerically. Unlike the work of Faisst and Eckhardt they considered a computational domain that was $50D$ long and so were able to capture localised turbulence (recall a typical puff is $20D$ in length). They initialised their runs in a similar manner to Peixinho and Mullin by taking an equilibrium puff from a higher Reynolds number computation. Because their domain was periodic in space they did not have to worry about the puff leaving the pipe and so could take measurements up to $Re = 1860$. From this they recovered similar results but with $Re_c = 1870$.

A final experimental paper (Hof et al., 2008) undertook the problem this time with significant improvement in the care taken (stricter temperature controls, automated measurements) which allowed them to increase observation times by 8 orders of magnitude. From this they concluded that there was no critical Reynolds number and were able to observe decaying puffs up to $Re = 2050$. With the expanded range of observable $Re$ they found evidence to suggest that the functional form of $\tau$ is superexponential - though they were unable to give a definitive form.

Recently in an attempt to bring together the numerical and experimental results Avila et al. (2010) combined the experimental work of Hof and the numerical work of Willis. This led to good agreement between the two parties that a superexponential scaling is indeed correct and perhaps an end to the debate. Certainly it fits in with recent results from Taylor-Couette flow where Borrero-Echeverry et al. (2010) found a similar superexponential law.

Whether a superexponential scaling is correct or not, and whatever form it may take, there is still no strong theoretical reason for why this should be the case. We will leave the lifetime debate at this juncture and will not consider it in this thesis.
The main thrust of theoretical work recently has been into the mechanisms which trigger turbulence and those which subsequently maintain it. Based on ideas from dynamical systems, turbulence has started to be viewed as being structured around an underlying skeleton of flow fields. Each of these flow fields is an exact solution of the Navier-Stokes equations. The full skeleton would be made up of large numbers of these solutions of varying forms and complexities, complemented by heteroclinic and homoclinic connections between them. These solutions give some sense to turbulence observed and dictate the nature of the turbulence. The simplest of these solutions in pipe flow are travelling wave solutions which we will discuss in Chapter 2 (the results of which were published in Pringle and Kerswell (2007)) and Chapter 3 (published in Pringle et al. (2009)). Chapter 4 introduces a more complicated form of exact solution - a relative periodic orbit. This chapter is a reproduction of Duguet et al. (2008a).

In the problem of transition, of particular interest is how a small perturbation can give rise to much more energetic turbulent flows. One approach to this is transient growth, a phenomenon well explored in the linear setting. We approach this problem using a method of direct numerical simulation (Chapter 5) before solving the full nonlinear problem of transient growth for one particular case in Chapter 6.
Chapter 2

Rotationally Asymmetric Travelling Waves

2.1 Introduction

2.1.1 Exact Solutions of the Navier-Stokes equations

In recent years two historically separate fields of theoretical research have started to lend ideas to each other - those of fluid dynamics and dynamical systems. It has long been recognised that a fluid can be thought of as a dynamical system, however it has a huge number of degrees of freedom associated. Typical dynamical systems studied would have perhaps less than ten degrees of freedom - beyond this being viewed as a large system. Strictly speaking a fluid has an infinite number of degrees of freedom, but by projecting it down onto, say, a set of basis functions (e.g Fourier modes, Chebyshev modes, etc) it can be approximated by a finite, but large, number of degrees of freedom. Supposing we were to use a Fourier approximation. The smallest length scale we would typically want to resolve is given by the Kolmogorov scale, \( l = Re^{-3/4} \). The highest Fourier number required then is given by \( 1/l \). Assuming we
are considering a three-dimensional flow the total number of degrees of freedom would be \( N = (Re)^3 \). This is still far too many to treat with the traditional techniques of dynamical systems, but some of the wider ideas can be taken.

One key idea is that before chaos can be supported in a dynamical system, exact non-trivial solutions of the underlying equations must exist. In terms of a fluid, before turbulence can be sustained exact non-(trivially-)laminar solutions of the Navier-Stokes equations must exist. The simplest form these could take would be of steady states solutions. As pipe flow has a mean flow along the pipe, it is likely that instead one would find solutions moving at a constant speed down the pipe - travelling waves.

The earliest attempts to calculate such solutions were by Smith and Bodonyi (1982). In this paper they present an asymptotic critical layer analysis that suggests the possibility of helical travelling waves with swirl. Later Landman (1990) attempted to calculate these waves numerically but found little indication of their existence. In a related problem, Nagata (1990) succeeded in numerically calculating modes in plane-Couette flow. To do this he introduced rotation. Rotating-Couette flow is known to be linearly unstable, and he traced out one of these bifurcations (a secondary solution). This he then examined for instabilities leading him to a tertiary solution. Once he had secured a tertiary solution he then continued it back to plane-Couette flow by smoothly decreasing the degree of rotation in a homotopic method. This idea was applied to pipe flow by Barnes and Kerswell (2000) who considered the related problem of flow in a rotating pipe and by Kerswell and Davey (1996) who looked at flow in an elliptical pipe. Neither proved sustainable in the limit of Hagen-Poiseuille flow.

By this time a more systematic approach was emerging. It had been recognised for some time (Kline et al., 1967; Smith and Metzler, 1983; Kim et al., 1987) that certain near-wall structures recurred frequently.
These typically took the form of regularly spaced streamwise vortices and streaks. The fundamental question is, are these structures a product of turbulence or is turbulence a product maintained by these structures? We must begin by asking how the flow can repeatedly undergo a cycle of rolls begetting streaks.

2.1.2 The Self-Sustaining Process

In order to capture self-sustaining structures in shear flows we will break the velocity perturbation field into three parts. Although much of the original work was performed in the scenario of plane-Couette flow, here we shall consider it within the framework of pipe flow. Here the laminar state will be referred to as $U_{\text{larm}} = U_{cl}(1 - s^2/s_0^2)$. The first of the velocity sub-fields represent streamwise rolls, and can be written as

$$
\begin{bmatrix}
U(s, \phi) \\
V(s, \phi) \\
0 \\
P(s, \phi)
\end{bmatrix}_{\text{rolls}}.
$$

(2.1.1)

For the time being we constrain ourselves to a fully linear approximation of the problem. The radial and azimuthal components of the Navier-Stokes equations give us the evolution of this form of perturbation as

$$
\begin{align*}
\partial_t U &= -\frac{1}{\rho} \partial_s P + \nu \nabla^2 U, \\
\partial_t V &= -\frac{1}{\rho} \partial_\phi P + \nu \nabla^2 V, \\
\partial_s (sU) + \partial_\phi V &= 0.
\end{align*}
$$

(2.1.2)

(2.1.3)

(2.1.4)

Clearly the underlying laminar profile does not impinge on any of these equations. This is no longer the case when we come to the axial component, as here it will interact with the radial component of the flow to
create the second sub field

\[
\begin{bmatrix}
0 \\
0 \\
W(s, \phi) \\
0
\end{bmatrix}_{\text{streaks}}
\]  

(2.1.5)

through the equation

\[
\partial_t W - \partial_s \left( U_{1am} \right) U = -\frac{1}{\rho} \partial_z P + \nu \nabla^2 W. 
\]  

(2.1.6)

In a linear setting any perturbation in pipe flow will, given sufficient time, decay due to viscosity as pipe flow is linearly stable at least at below \( Re = 10^7 \) (Meseguer and Trefethen, 2003). Further to this, even if nonlinear effects are included, then any perfectly streamwise independent perturbation must also decay. This might seem problematic, but happily we can circumvent this difficulty. Streaks correspond to local patches in the streamwise velocity field that are going faster or slower than the fluid surrounding it. This can lead to an inflection point and Rayleigh’s Inflection Point Theorem tells us that this can lead to instability.

Zikanov (1996) analysed this in pipe flow and demonstrated that given sufficiently energetic rolls then the resultant streaks would indeed develop inflectional instabilities. This leads us to the third and final constituent of the velocity field

\[
\begin{bmatrix}
\hat{u}(s, \phi, z) \\
\hat{v}(s, \phi, z) \\
\hat{w}(s, \phi, z) \\
\hat{p}(s, \phi, z)
\end{bmatrix}_{\text{waves}}
\]  

(2.1.7)

which captures the key ‘feedback’ effects. Our velocity field can now be
written in full as
\[
\begin{bmatrix}
  u \\
  v \\
  w \\
  p \\
\end{bmatrix}_{\text{rolls}} +
\begin{bmatrix}
  0 \\
  0 \\
  W(s, \phi) \\
  0 \\
\end{bmatrix}_{\text{streaks}} +
\begin{bmatrix}
  \hat{u}(s, \phi, z) \\
  \hat{v}(s, \phi, z) \\
  \hat{w}(s, \phi, z) \\
  \hat{p}(s, \phi, z) \\
\end{bmatrix}_{\text{waves}} =
\begin{bmatrix}
  U(s, \phi) \\
  V(s, \phi) \\
  0 \\
  P(s, \phi) \\
\end{bmatrix}
\]

This full velocity and pressure field must satisfy the entirety of the Navier-Stokes equations rather than just the cut down parts we have considered so far. If we consider the rolls and streaks as being \((U, P)\) and the waves as \((\hat{u}, \hat{p})\) the equations to be satisfied are
\[
\frac{\partial}{\partial t}(U + \hat{u}) + U_{\text{lam}} \frac{\partial}{\partial z}(U + \hat{u}) + (U + \hat{u}) \frac{\partial}{\partial s}U_{\text{lam}}e_z + (U + \hat{u}).\nabla(U + \hat{u})
= -\frac{1}{\rho} \nabla\hat{p} + \nu \nabla^2(U + \hat{u}),
\quad (2.1.9)
\]
\[
\nabla \cdot (U + \hat{u}) = 0.
\quad (2.1.10)
\]

Subtracting away equations (2.1.2)-(2.1.4) and (2.1.6) then leaves us with the remaining equations for the waves to satisfy as being
\[
\frac{\partial}{\partial t} \hat{u} + U_{\text{lam}} \frac{\partial}{\partial z} \hat{u} + \hat{u} \frac{\partial}{\partial s}U_{\text{lam}}e_z + (U + \hat{u}).\nabla(U + \hat{u})
= -\frac{1}{\rho} \nabla\hat{p} + \nu \nabla^2 \hat{u},
\quad (2.1.11)
\]
\[
\nabla \cdot \hat{u} = 0.
\quad (2.1.12)
\]

Linearising (2.1.11)-(2.1.12) gives the linear stabilitity problem for a disturbance to rolls/streaks formation as described by \(U\).

So far we have described how rolls can linearly create streaks and how to determine whether the streaks are unstable to wave like perturbation. If they are, then we also have provided the necessary equations to model them too.

Hamilton et al. (1995) were the first to suggest a cycle along these lines. Their cycle ran as streamwise vortices create streaks via linear
advection. The streaks exhibit instability to create $z$-dependent flow. Unspecified nonlinear interactions then regenerated the original rolls. This unspecified nonlinear interaction is the key to whether this process can work or not. One of the neat aspects of this formulation is that as soon as one chooses a roll structure $\mathbf{u} = (U, V, 0)$ then the remainder of the steps are predescribed. The streaks follow immediately as do their instability and the resultant waves. We can thus compare the initial rolls to the effects of the wriggling.

To do this we must rethink equations (2.1.2)-(2.1.4) a little more precisely. If we had initially included the full velocity field $\mathbf{u} = U + \tilde{u}$ and considered the streamwise average of the Navier-Stokes equations we would have found

$$
\partial_t U + \mathbf{e}_x \cdot (U \nabla U + \bar{u} \nabla \bar{u}) = -\frac{1}{\rho} \partial_x P + \nu \nabla^2 U, \quad (2.1.13)
$$

$$
\partial_t V + \mathbf{e}_\phi \cdot (U \nabla U + \bar{u} \nabla \bar{u}) = -\frac{1}{\rho} \partial_\phi P + \nu \nabla^2 V, \quad (2.1.14)
$$

$$
\partial_x (sU) + \partial_\phi V = 0, \quad (2.1.15)
$$

where here $\langle \cdot \rangle$ represents the streamwise average defined by

$$
\langle \cdot \rangle := \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \cdot \, dz. \quad (2.1.16)
$$

This is the same equation as before but includes both the nonlinear term from the rolls and a streamwise averaged version of the nonlinear term from the waves. From this we can see that it is $\bar{u} \nabla \bar{u}$ which will govern the feedback onto the initial rolls. By making this comparison, Waleffe (1997) managed to show choosing rolls that corresponded to those which exhibit the slowest decay due to viscosity corresponded to good feedback, thus (heuristically at any rate) closing the cycle. Using this comparison to pin-point likely starting points in phase space he then succeeded in capturing such waves in plane-Couette flow (Waleffe, 1998).
2.1.3 An Algorithm for finding Travelling Waves

In this section we present a method for calculating travelling waves. These waves represent exact numerical solutions of the Navier-Stokes equations for whatever problem is being considered - here pressure driven pipe flow, but in principle any shear flow. The method is that suggested by Waleffe (1998), and is in essence a constructive homotopy. A specifically designed related problem is solved and then continuation methods are used to connect alternative solutions in the new problem back to the desired one.

For the following method we work at a fixed Reynolds number. The first step is to choose a forcing function which will create axially-aligned rolls in the pipe. The forcing can be chosen as desired and is not pre-described by the self-sustaining process itself. However, an arbitrary choice is unlikely to work and so some degree of a priori knowledge is required.

We now consider the problem of pipe flow with the forcing function added. If the amplitude of the forcing is sufficiently small, clearly the resulting flow will be close to the usual Hagen-Poiseuille flow and linearly stable. If large in amplitude the flow will become unstable. This leads to the natural conclusion that there must exist a critical forcing amplitude which corresponds to a neutrally stable flow configuration - a bifurcation point.

The bifurcation point will have with it an associated wavelength and frequency. The stability analysis requires one to look for instabilities in the form of a single axial Fourier mode and so the axial wavenumber provides a degree of freedom to be chosen beforehand. The frequency gives the phase speed of the travelling wave. Out of the bifurcation point a secondary solution emerges which we can write in the usual form

\[ u(s, \phi, z, t) = u(s, \phi, z - ct) \]  \hspace{1cm} (2.1.17)
As well as rotational symmetry, it is also possible to impose additional symmetries on the flow. There are three simple symmetries that can be explored.

\[ S : (u, v, w, p)(s, \phi, z) \rightarrow (u, -v, w, p)(s, -\phi, z + \pi/\alpha) \] (2.1.18)

\[ Z_\psi : (u, v, w, p)(s, \phi, z) \rightarrow (u, -v, w, p)(s, 2\psi - \phi, z) \] (2.1.19)

\[ \Omega_{m_0} : (u, v, w, p)(s, \phi, z) \rightarrow (u, v, w, p)(s, \phi + \pi/m_0, z + \pi/\alpha). \] (2.1.20)

The first of these is shift-&-reflect symmetry. As all travelling waves are axially periodic, this links the form in the first half of the wavelength to that in the second half. Specifically it says that the second half of the wave looks like a reflection of the first half in the \( \phi = 0 \) axis. \( Z_\psi \) represents reflection in the \( \phi = \psi \) axis. The last symmetry, \( \Omega_{m_0} \), is shift-&-rotate. Similarly to \( S \) it links the first and second halves of an axial wavelength together, but this time through a rotation rather than a reflection. The three forms of symmetry are not truly independent, as

\[ S \Omega_{m_0}^j = Z_{j\pi/m_0} \quad j = 1, 3, 5, \ldots, 2m - 1. \] (2.1.21)

The superscript \( j \) is used to indicate that \( \Omega_m \) should be applied \( j \) times.

While there is no reason to assume travelling waves would possess any of these symmetries, they do lead to large reductions in computational storage. If we impose the desired symmetry in the stability calculation, then any travelling wave found as a result will automatically possess the same symmetry.

We now have three degrees of freedom associated with the solution - the Reynolds (\( Re \)), the axial wavenumber (\( \alpha \)), and the phase speed (\( c \)). These three quantities are interrelated and any one of them can be written as a function of the other two. By allowing these quantities to vary it is possible to continue the new secondary solution through phase
space. In this way we hope to be able to reduce the forcing amplitude smoothly by compensating with, for instance, the phase speed in the hopes of returning to zero-forcing. This would then represent a new solution of the original problem that was considered.

2.1.4 Known Travelling Waves in Pipe Flow

After the success that Waleffe had in applying this to plane-Couette flow, two groups applied the same methodology to pipe flow. First to be published was a letter by Faisst and Eckhardt (2003). A second paper Wedin and Kerswell (2004) examined the solutions in rather more detail, and provided a fuller set of solutions, along with a more detailed description of how they can be calculated.

Wedin & Kerswell’s Numerical Procedure

Wedin and Kerswell considered the flow through a pipe of constant circular radius $s_0$ of a Newtonian incompressible fluid with constant density $\rho$ and constant kinematic viscosity $\nu$. The fluid was forced through the pipe by a constant pressure gradient of

$$\nabla p^* = -\frac{4\rho W^*}{s_0^2}e_z,$$  \hspace{1cm} (2.1.22)

corresponding to the laminar state

$$u^* = U_{cl}\left(1 - \frac{s^2}{s_0^2}\right)e_z.$$  \hspace{1cm} (2.1.23)

The asterisks represent dimensional quantities. These were non-dimensionalised using the pipe radius ($s_0$) and the centreline velocity of the laminar state ($U_{cl}$). It is easier, at least computationally, to consider the ‘perturbation’ velocity field. Considering the full velocity field as $u = U + \tilde{u}$, where $U(s) = (1 - s^2)e_z$ is the laminar state, then the resulting equations for
the perturbation field $\tilde{u}$ in cylindrical coordinates of $(s, \phi, z)$ are

$$\frac{\partial \tilde{u}}{\partial t} + (1 - s^2) \frac{\partial \tilde{u}}{\partial z} - 2s \tilde{u} e_z + \tilde{u} \cdot \nabla \tilde{u} = -\nabla p + \frac{1}{Re} \nabla^2 \tilde{u} + \frac{4}{Re} e_z,$$

(2.1.24)

$$\nabla \cdot \tilde{u} = 0,$$

(2.1.25)

subject to the standard no-slip boundary condition

$$\tilde{u}(1, \phi, z) = 0.$$  

(2.1.26)

The Reynolds number above is given by $Re = U_{cl} s_0 / \nu$. At this point we note that there is a second Reynolds number that is frequently used, especially among the experimental community. This is based upon the mass-flux along the pipe and is defined by

$$Re_m := \frac{2s_0 \overline{W}}{\nu}$$  

(2.1.27)

where

$$\overline{W} := \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \tilde{u} e_z s ds d\phi$$  

(2.1.28)

is the bulk velocity. In the case of laminar flow $Re = Re_m$, however once turbulence has set in the two values will diverge. The extent to which they change can be used as a measure of the level of turbulence.

In order to calculate exact solutions of the Navier-Stokes equations one does not need to perform a traditional numerical simulation. Instead the method used focuses on writing the problem as a large set of simultaneous equations which are solved numerically with a Newton-style solver. The first step to this end is to create a spectral representation of the flow. Fourier modes are used in the two periodic directions - azimuthal and axial - while Chebychev polynomials are used radially. Spectral approaches to pipe flow are traditionally hindered by a need for care at the axis of
the pipe. Typically this is treated with a regularity condition at \( s = 0 \)

\[
\lim_{s \to 0} \frac{\partial u}{\partial \phi} = 0 \quad \text{and} \quad \lim_{s \to 0} \frac{\partial p}{\partial \phi} = 0.
\] (2.1.29)

Instead, here we take care of it by considering the domain \( \{ -1 \leq s \leq 1, 0 \leq \phi \leq \pi \} \) rather than \( \{ 0 \leq s \leq 1, -\pi \leq \phi \leq \pi \} \). Symmetries relate the solution in \( -1 \leq s < 0 \) to that in \( 0 < s \leq 0 \), so we restrain the necessary collocation points to the positive half of the \( s \)-axis.

The no-slip boundary condition is built into the radial basis used. Chebychev polynomials \( T_n(s) \) all evaluate to 1 at \( s = 1 \) and alternate between +1 and −1 at \( s = -1 \). Combining pairs of Chebychev polynomials leads to a basis which will always take a value of 0 at \( s = \pm 1 \).

Thus we can write our solution as

\[
\begin{bmatrix}
\ddot{u} \\
\ddot{v} \\
\ddot{w} \\
\ddot{p}
\end{bmatrix} = \sum_{l=-L}^{L} \sum_{m=-M}^{M} \sum_{n=0}^{N-1} \begin{bmatrix}
\ddot{u}_{mln} T_n(s; mm0) \\
\ddot{v}_{mln} T_n(s; mm0) \\
\ddot{w}_{mln} T_n(s; mm0) \\
\ddot{p}_{mln} T_n(s; mm0)
\end{bmatrix} e^{imm_0 \phi} e^{i ln(x-ct)},
\] (2.1.30)

where radial functions are defined by

\[
\Theta_n(s; i) := \begin{cases} 
T_{2n+2}(s) - T_{2n}(s) & \text{i odd}, \\
T_{2n+3}(s) - T_{2n+1}(s) & \text{i even},
\end{cases} \quad \text{(2.1.31)}
\]

\[
\Phi_n(s; i) := \begin{cases} 
T_{2n+3}(s) - T_{2n+1}(s) & \text{i odd}, \\
T_{2n+2}(s) - T_{2n}(s) & \text{i even},
\end{cases} \quad \text{(2.1.32)}
\]

\[
\Psi_n(s; i) := \begin{cases} 
T_{2n+1}(s) & \text{i odd}, \\
T_{2n}(s) & \text{i even},
\end{cases} \quad \text{(2.1.33)}
\]

and

\[
T_n(x) := \cos(n \arccos x), \quad \text{(2.1.34)}
\]
The quantities $\alpha, m_0$ and $c$ are, respectively, the axial wavenumber (can take any positive value) the azimuthal wavenumber (must be a non-negative integer) and the phase speed of the travelling wave.

Demanding that the solution must be real leads to a reduction in the number of independent coefficients via the relationship

$$x_{ml} = \bar{x}_{m-l}$$

(2.1.35)

where $x$ is one of $u, v, w$ or $p$ and $\bar{\cdot}$ here represents the complex conjugate. The symmetries (2.1.18)-(2.1.20) further reduce the number of coefficients. Both $S$ and $Z$ give a relationship between $x_{ml}$ and $x_{-ml}$.

For shift-&-reflect symmetry we find that

$$\tilde{u}_{-ml} = (-1)^l \tilde{u}_{ml},$$
$$\tilde{v}_{-ml} = -(1)^l \tilde{v}_{ml},$$
$$\tilde{w}_{-ml} = (-1)^l \tilde{w}_{ml},$$
$$\tilde{p}_{-ml} = (-1)^l \tilde{p}_{ml},$$

(2.1.36)

while for $Z$–symmetry we have

$$\tilde{u}_{-ml} = \tilde{u}_{ml},$$
$$\tilde{v}_{-ml} = -\tilde{v}_{ml},$$
$$\tilde{w}_{-ml} = \tilde{w}_{ml},$$
$$\tilde{p}_{-ml} = \tilde{p}_{ml}.$$  

(2.1.37)

Shift-&-rotate symmetry is slightly different to the previous two as instead of relating coefficients of $\pm m$, it instead simply eliminates half the coefficients as it states that

$$x_{ml} \equiv 0 \quad \text{if} \quad m + l \text{ is even}.$$  

(2.1.38)

Having chosen a spectral formulation, the equations are then evaluated and solved at the collocation points given by the positive zeroes of
One additional equation must also be solved. This is to remove degeneracy in the phase speed \( c \) and is given by

\[
\Im \left\{ \sum_{n=0}^{N-1} u_{-11n} \Theta_n(0.1; -m_0) \right\} = 0. 
\] (2.1.39)

By considering each Fourier-mode at each collocation point we have an algebraic nonlinear system of equations. The unknown quantities which must be solved for are the spectral coefficients, the Reynolds number, the phase speed, the axial wave number and the forcing amplitude. As we do not have as many equations as unknowns, we must keep some of the quantities fixed. To this end we will always keep two of the quantities \( (Re, c, \alpha, A) \) fixed, where \( A \) is the forcing amplitude. The equations can be written as

\[
\mathcal{F}(\bar{u}_{min}, \bar{v}_{min}, \bar{w}_{min}, \bar{p}_{min}; c, Re, \alpha, A) = 0. 
\] (2.1.40)

The system is solved using the package \textsc{PITCON} created by Rheinboldt & Burkhardt. This is a branch following routine based upon Newton-Raphson iteration. Each solution of (2.1.40) is an exact numerical solution of the Navier-Stokes. By ‘exact’ we mean that the solution can be found to arbitrary accuracy.

The Discovered Travelling Waves

As already emphasised, the choice of roll structure is critical to the success of the algorithm for calculating travelling waves. In keeping with the suggestions of Waleffe, Wedin and Kerswell chose the rolls that would survive for the longest period of time against viscous decay (Faisst and Eckhardt simplified matters by choosing a forcing which resembled the rolls of Waleffe’s work). This allows the rolls to remain active for as long as possible, generating the strongest streaks and giving the best chance of the instability setting in, in time to reinvigorate the rolls. This
Table 2.1: This table contains the eigenvalues for the least decaying streamwise rolls. They are named in the form $\lambda_{m_0n}$ where $m_0$ is the azimuthal periodicity and $n$ is the number of zeros in the radial component of the velocity field.

<table>
<thead>
<tr>
<th>$m_0$</th>
<th>$\lambda_{m_01}$</th>
<th>$\lambda_{m_02}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.135622302</td>
<td>8.417244140</td>
</tr>
<tr>
<td>2</td>
<td>6.380161896</td>
<td>9.761023130</td>
</tr>
<tr>
<td>3</td>
<td>7.588342435</td>
<td>11.06470949</td>
</tr>
<tr>
<td>4</td>
<td>8.771483816</td>
<td>12.33860420</td>
</tr>
<tr>
<td>5</td>
<td>9.936109524</td>
<td>13.58929017</td>
</tr>
<tr>
<td>6</td>
<td>11.08637002</td>
<td>14.82126873</td>
</tr>
</tbody>
</table>

The roll structure is found by considering a linearised version of the streamwise and azimuthal components of the (streamwise averaged) Navier-Stokes equations:

$$
\partial_s U(s, \phi, t) = -\partial_s P(s, \phi, t) + \frac{1}{Re} \nabla^2 U(s, \phi, t) \quad (2.1.41)
$$

$$
\partial_t V(s, \phi, t) = -\frac{1}{s} \partial_\phi P(s, \phi, t) + \frac{1}{Re} \nabla^2 V(s, \phi, t) \quad (2.1.42)
$$

$$
\partial_s (sU) + \partial_\phi (V) = 0. \quad (2.1.43)
$$

Solving the derived eigenvalue problem gives a roll structure of the form

$$
U = [J_{m_01}(\lambda s) + J_{m_0-1}(\lambda s) - J_{m_0-1}(\lambda)s^{m_0-1}] \cos m_0 \phi \quad (2.1.44)
$$

$$
V = [J_{m_01}(\lambda s) - J_{m_0-1}(\lambda s) + J_{m_0-1}(\lambda)s^{m_0-1}] \sin m_0 \phi \quad (2.1.45)
$$

where $\lambda$ (the eigenvalue) is given from the no-slip requirement as $J_{m_01}(\lambda) = 0$. Here $J$ is the Bessel function of the first kind and $m_0$ gives the azimuthal periodicity of the rolls.

The eigenvalue problem solved above gives a whole family of solutions described by two parameters - the azimuthal wavenumber and the number of zeroes in the radial component of the flow field ($U$). These are all defined by the parameter $\lambda_{m_0n}$, summarised in table 2.1.

In the two papers, six separate travelling waves were identified and
Figure 2.1: From left to right, top to bottom: S2, S3, S4, S5, S6, U1. All slices have the same contour levels running from $-0.19W$ (dark streaks) to $+0.19W$ (light streaks) in units of $0.02W$. All are lower-branch solutions and correspond to values of $(Re_m, \alpha)$ of: S2 (2400, 1.55), S3 (2400, 2.5), S4 (2400, 3.125), S5 (3000, 4.11), S6 (3000, 4.73) & U1 (3046, 2.17).

explored. Five of these came from using $\lambda = \lambda_{21}, \lambda_{31}, \ldots, \lambda_{61}$. These travelling waves all possessed S symmetry. Despite extended effort on the part of Wedin and Kerswell, the Z symmetric counterparts were not found (Wedin, 2004).

The final travelling wave that was discovered was markedly different to the others previously found. In an attempt to find a travelling wave that was not rotationally symmetric, Wedin and Kerswell adopted a slightly different approach. They first imposed the forcing corresponding to $\lambda_{22}$. Although this had two-fold rotational symmetry they looked for an azimuthally subharmonic instability to the forced flow. That is to say an instability with half the azimuthal wavenumber $- m_0 = 1$. This they succeeded in doing, but only for a much higher Reynolds number. The consequent travelling wave then had not only the S symmetry previously seen, but instead all three symmetries as described by (2.1.21).
Table 2.2: Here is an overview of all the originally discovered travelling waves. The naming convention is such that the letter gives the axial symmetry and the number gives the rotational symmetry. The first four listed travelling waves (S2-S5) were reported by both Faisst and Eckhardt (2003) and Wedin and Kerswell (2004). The final two were only calculated by the latter publication.

The physical appearance of S2-S6 are all very similar (figure 2.1) consisting of a ring of alternating fast and slow streaks with intermittent rolls. The unclassified travelling wave is of a rather different form, in keeping with it’s peculiar symmetries.

Having identified six different travelling waves the next issue was to explore their location in phase space. The primary aim was to find the value of $\alpha$ which gave the minimum $Re$ at which each travelling wave. At this optimising value of $\alpha = \alpha^*$ the travelling wave was traced out in $(Re, c)$–space. The travelling waves discovered are summarised in table 2.2.

While $\alpha$ and $c$ are natural parameters of the travelling waves, they are perhaps not the most physically meaningful quantities to consider. To this end we introduce the quantity friction factor. Turbulence leads to an increase in wall shear stress. In order to maintain a given mass flux an increase is required in the driving pressure gradient. This increase is measured by the friction factor

$$\Lambda := -\frac{1}{\rho} \frac{dp}{dz} \sqrt{\frac{U^2}{2D}} = \frac{64Re}{Re_m^2}$$

For the laminar case we simply have $\Lambda = 64/Re$. Once a turbulent phase is entered the mass flux (and thus the mass Reynolds number) will drop
Figure 2.2: A plot of phase speed (in units of $\overline{W}$) against the flux Reynolds numbers of all the travelling waves described by Wedin and Kerswell. Each travelling wave is shown for its own value of $\alpha^*$ (see table 2.2). The numbers correspond to the rotational symmetry of each branch.

and so the friction factor increases with $64/Re$ acting as a lower bound. The higher the friction factor the higher, in some sense, the degree of turbulence.

The travelling waves are plotted to show their variation in phase space with Reynolds number both in terms of their phase speed (figure 2.2) and their friction factor (figure 2.3). The latter plot is particularly revealing. We can clearly see that there are two branches for each travelling wave - a lower branch and an upper branch. The lower branch is ‘closer’ to the laminar, at least in terms of the additional wall stress, while the upper branch is close to turbulent observations. This suggests that the upper branch may be relevant to actual turbulent dynamics, while the lower branch is more likely important to turbulence transition.

It should be briefly noted as well that the branches are ‘switched’
Figure 2.3: A plot of friction factor, $\Lambda$, against the flux Reynolds numbers of all the travelling waves described by Wedin and Kerswell. Each travelling wave is shown for its own value of $\alpha^*$ (see table 2.2) and the colours correspond to those in figure 2.2. The blue dashed line is the laminar state while the blue dash-dot line is a parameterisation of experimental data.
between figures 2.2 and 2.3. That is to say that the branch with the lower friction factor has a higher phase speed. The terms lower and upper branch come from the latter diagram as it appears to have greater physical relevance. Inspection of such quantities as perturbation energy and energy dissipation lead to the same definitions of the lower and upper branches.

Since the discovery of these travelling wave solutions experimentalists have been able to observe closely matching structures in the laboratory. Hof et al. reported these findings in 2004 and noted that while an arbitrarily chosen snapshot of turbulent flow bore little resemblance to any given travelling wave, it was possible to pick moments in the flow where close similarities occurred (see figure 2.4). From this it was argued that travelling waves could act as a skeleton around which actual turbulence was based.

Further work by Kerswell and Tutty (2007) and Willis and Kerswell (2008) looked at numerical simulations of pipe flow and attempted to quantify the frequency and nature of these close approaches to travelling waves. Both concluded that around 10% of the time, turbulence could in some way be said to approach a known solution. Furthermore, the most frequently visited waves had either two-fold or three-fold rotational symmetry (the first to appear in terms of Reynolds number), and that it generally happens during the transitions to and from turbulence. A third paper which addressed this problem was Schneider et al. (2007a). This paper looked at a short pipe (like Kerswell and Tutty) but addressed the issue that there are undoubtably numerous unknown exact solutions. They noted that such structures are dominated by an arrangement of cross-stream rolls and streamwise streaks. They therefore looked for this pairing rather than for known solutions and concluded the frequency visits is around 20 – 24%. One might expect this to provide a reasonable upper-bound on the frequency of visits, however unfortunately they also required the flow field to be rotationally symmetric. Because of this,
Figure 2.4: Here we compare snapshots of experimental flow through a pipe with the numerically calculated travelling wave solutions. The contours show the axial velocity of the perturbation (red fast streaks, blue slow ones), while the arrows are the in plane \((s, \phi)\) velocities. The upper row is the experimental data taken at the Reynolds numbers \(Re = 2000\) (A), \(Re = 2500\) (C) and \(Re = 5300\) (E). The lower row is the numerically calculated travelling waves with rotational symmetry \(R_3\) (B), \(R_2\) (D) and \(R_6\) (E). The plot is taken from Hof et al. (2004).
their result is neither an upper, nor a lower, bound.

### 2.1.5 The Laminar/Turbulent Boundary

For many years experimentalists and theoreticians have been intrigued by what criteria determines whether a perturbation will trigger (transient) turbulence or smoothly relaminarise. The fact that pipe flow is linearly stable leads to the observation that, if the Reynolds number is sufficient, either result is possible and dependent on the form and amplitude of the perturbation.

If we fix the form of the perturbation, then whether transition is seen must depend only upon the amplitude of the perturbation. Large amplitudes should lead to turbulence, while small amplitudes to relaminarisation. One can imagine an intermediate amplitude that falls exactly between these two possibilities. Taking this as the initial condition would lead to the flow evolving in a chaotic manner but remaining of intermediate energy - never reaching either saturated turbulence or relaminarising.

Let us consider a phase space consisting of every possible initial condition for a pipe flow experiment, with the laminar case as the origin. Figure 2.5 shows a two dimensional illustration of it. The greater the amplitude of an initial condition, the further it is from the origin. A boundary of varying amplitude is theorised which separates initial conditions that will lead to turbulence from those that will lead to relaminarisation. This is the so-called laminar/turbulent boundary or edge of chaos.

The first to make use of this were Itano and Toh (2001) who were studying plane-Poiseuille flow. They examined direct numerical solutions for a period of quasi-steady flow. From this they then attempted to shoot along the boundary to a nearby travelling wave. The full implications of this concept were expounded upon later by Skufca et al. (2006) and Schneider et al. (2007b). From this work it became apparent that edge trajectories were strongly chaotic, but could at times seem to evolve
Figure 2.5: A two dimensional representation of phase space. The dash-dot line represents the laminar/turbulent boundary. Initial conditions (black dots) inside the boundary decay back to the laminar state (grey dot) while those outside go off into turbulence. Two conditions close to each other but either side of boundary can show initially similar evolution before diverging.
slightly more steadily. Taking a time average of the flow during these periods led to a well defined cross-sectional profile reminiscent of the travelling waves found by Faisst and Eckhardt and Wedin and Kerswell (figure 2.6).

2.2 A New Form of Forcing

2.2.1 The Forcing Function

The similarity between the time-averaged edge trajectory and the originally discovered travelling waves in pipe flow suggest that there is likely to be an underlying travelling wave responsible for the tidy structure observed. Wedin and Kerswell did attempt to find a non-rotationally symmetric travelling wave (other than the highly-symmetric, unclassified travelling wave). To this end they used the roll structure derived in section 2.1.4. This gives two large rolls occupying half the pipe each and they met with no success. Forearmed with the calculation of Schneider...
et al., we instead created a forcing function with a localised pair of rolls. This requires distributing the energy across a band of azimuthal wave number. The radial velocity field is then described by

\[ u(s, \phi) \propto \Re \{ e^{-1/s} (1 - s^2) \sum_{m=1}^{7} [1 + \cos(m \frac{\pi}{7})] e^{im\phi} \} \quad (2.2.47) \]

and the azimuthal component simply taken to maintain incompressibility. The rolls that this produces is illustrated in figure 2.7.

Along with prescribing a forcing function, one must also choose the Reynolds number at which to work and the wavelength that the instability analysis is performed at. The choice of Reynolds number is important, as the larger it is the more unstable the flow (and so the smaller the forcing amplitude required). Increasing the Reynolds number is not always a good idea, however, as it becomes harder to resolve correctly and so the calculation is slower. To this end we used an initial Reynolds number of 4000 - comfortably above the point at which we can expect the travelling wave to appear but not so high as to cause resolution issues. The choice of wavelength is a little freer and was chosen to minimise the initial forcing amplitude required to reach neutral stability.
Without doing an exhaustive search we settled for $\alpha = 0.75$.

### 2.2.2 Route to Zero Forcing

Having decided on a forcing function we then followed the method previously described to try and trace the bifurcation back to the point of no forcing. We recall that at the point of bifurcation the flow is ‘laminar’ and there is no perturbation energy.

For this work we used the inherited codes previously used by Wedin and Kerswell (2004). These had to be adapted to deal with generalised forcing functions, solutions that were no longer symmetric or that possessed extra degrees of freedom (2.3.4).

As soon as we move away from the bifurcation then we hope to leave the laminar state and so the quantity

$$||u||_{3d} := \sqrt{\sum_{n=0}^{N-1} |u_{11n}|^2 + |v_{11n}|^2 + |w_{11n}|^2}$$

(2.2.48)

is useful in determining ‘how far’ from the laminar state we are. As was previously eluded to, we have four main parameters at our disposal - $Re, c, \alpha$ & $A$ - and in the smooth continuation method we always keep two of these fixed. One of the remaining parameters is then changed and the final parameter is allowed to vary to compensate for the one that was changed.

To begin with we kept $Re$ and $\alpha$ fixed and attempted to reduce $A$ by allowing $c$ to compensate. This was somewhat ineffective. Figure 2.8 shows the path taken. The initial bifurcation was traced out and although subcritical in $A$ it extended less than half the distance back to the point of zero forcing before turning back to higher amplitudes. The point of minimal forcing from this loop was taken as the starting point for the next phase. Here $c$ was fixed and instead $Re$ was allowed to vary. Increasing the Reynolds number unsurprisingly led to a straightforward
Figure 2.8: A plot of energy of the perturbation against the amplitude of the forcing applied. Where the line crosses the zero forcing axis with non-zero energy, a new finite amplitude solution is found. The solid lines represent where the phase speed was allowed to vary, and the dashed line is where the Reynolds number was varied.

decrease in the forcing required. However it was not apparent that zero forcing would ever be reached and as higher Reynolds number requires higher spectral truncation this method was curtailed at $Re = 8137$. A return to allowing the phase speed to vary at this higher Reynolds number led to a successful return to zero forcing.

2.3 A New Family of Travelling Waves

2.3.1 The Mirror-Symmetric Travelling Wave

Our initial motivation was to find a rotationally asymmetric travelling wave in order to complete the set of waves originally discovered and also (more pertinently) to attempt to find an underlying mode fundamental to the laminar/turbulent boundary. In actuality we discovered a connected family of travelling waves, the backbone of which is the mirror-symmetric
Figure 2.9: Two cross-sectional slices of the mirror-symmetric travelling wave. On the left is a slice at one instance in time while on the right is a time average. The left and right halves of the pipe are clearly mirror images of each other. The contour levels in both plots are at intervals of $0.02W$ from $-0.19W$ (dark) to $+0.19W$ (light).

Similarly to $U_1$ (table 2.2), this wave possesses all of the axial symmetries as described by equation 2.1.21. Unlike $U_1$, this travelling wave has four clear fast streaks around the edge, and three slow streaks across the middle. Rolls run in between the fast and slow streaks. These rolls come in two pairs - an upper and a lower pair - which as one goes down the pipe take it in turn to dominate. This can all be clearly seen in figure 2.9.

2.3.2 The Asymmetric Travelling Wave

All known travelling waves in pipe flow are unstable, and as such any stability analysis that is performed will reveal at least one unstable eigenvalue with corresponding eigendirection (indeed, for all known cases there are at least two unstable directions - as long as one looks in the full phase space. If one just considers phase space with the same degree of rotational symmetry as the wave then it is possible to have just one unstable direction). As one increases the Reynolds number it is possible
for the number of unstable directions to increase. This is hardly surprising, although what is worth noting is that this appears to be an unusual occurrence. For a new unstable direction to appear, the associated eigenvalues must cross from having negative real part to a positive real part, and at this point there is a bifurcation. In general the imaginary part will be non-zero. This gives the secondary structure a frequency relative to the underlying travelling wave. However, if the imaginary part is zero then the structure that bifurcates off will be another travelling wave, which can be traced out through phase space in the same way.

Previously this had been merely a theoretical possibility. None of the previously found travelling wave solutions in pipe flow (or indeed in any of the other studied shear flows) have at any point an exactly zero eigenvalue. However, the asymmetric solution originally looked for turned out to branch off of the mirror-symmetric wave in exactly this way.

This bifurcation breaks the mirror-symmetry of the solution, and so the asymmetric wave only retains the shift-&-reflect symmetry. The change in structure occurs rapidly, over a period of only a few decades of Reynolds number. The rolls on one side of the pipe start to dominate over the other and rapidly the structure changes so that one side of the pipe retains two rolls while the other is relatively untouched. This is all illustrated in figure 2.10. Of course, there is no reason why one side of the pipe should dominate over the other and so there are two possible solutions, mirror images of each other. In all macroscopic quantities (total perturbation energy, total wall shear stress, etc.) they are indistinguishable and the only difference is visual.

The asymmetric wave looks a lot like half of the mirror-symmetric wave, perhaps unsurprisingly. There are two fast streaks located at the edge of the pipe on one side, and a single slow streak sandwiched between them, though slightly more centrally located. As already mentioned these are accompanied by two rolls on the same side of the pipe as the
Figure 2.10: The symmetry breaking bifurcation through which $S1$ appears. The contour levels in all the cross-sections are the same. The Reynolds number increases in increments of 5 from $Re = 1770$ at the top left (the point of bifurcation) through to $Re = 1810$ at the bottom right. The contour levels are the same as in figure 2.9.
streaks. As you progress down the pipe, first one roll dominates, and then the other. Again, all this is shown in Fig. 2.11. In keeping with the previous results we will refer to the asymmetric travelling wave as S1.

### 2.3.3 Phase Space

In exploring the new travelling waves, the first task was to find the value of $\alpha$ which corresponded to minimising the Reynolds number at which the mirror-symmetric travelling wave appeared. This was done by tracing initially fixing $Re_m$ and allowing $c$ and $\alpha$ to vary. This gives closed loops which shrink in size as the saddle node is approached as evidenced in figure 2.12.

This figure illustrates a few points very starkly. Firstly we note that the minimum Reynolds number is very low indeed. Previously the lowest Reynolds number at which any travelling wave was known to exist was S3 at $Re = 1251$ (table 2.2). The mirror-symmetric mode exists as low
Figure 2.12: Phase velocity $C$ in units of the mean axial velocity as a function $\alpha$. The solid lines correspond to the mirror-symmetric travelling wave while the dashed lines are the asymmetric mode. The numbers correspond to the fixed $Re$ for each loop.
as $Re = 773$, while the asymmetric mode appears to continue to around $Re = 820$. That these travelling waves exist at such low Re is not, in and of itself, important. However empirical evidence seems to suggest that the travelling waves which appear at the lowest Reynolds numbers are those that are most frequently visited during turbulence. This could suggest that both S1 and the asymmetric travelling wave may play a key role in turbulence.

The way in which S1 bifurcates is also clearly shown, and it is notable that as it moves to higher Reynolds number, it begins to become slightly more tangled and complicated in phase space. Finally, we note that the optimal $\alpha$ to minimise $Re$ for the mirror-symmetric travelling wave is $\alpha^* = 1.44$. This is significantly away from the value of $\alpha$ which minimised the forcing amplitude needed for a bifurcation.

We also wish to explore how the travelling waves behave as the Reynolds number is increased. To do this we set a fixed value for $\alpha$ and allowed $c$ to vary to compensate for the increasing $Re$. We explored it at two different values of $\alpha$ being 0.75 and 1.44 - the initially used value and the minimising value. The large gulf between when the new waves appear and when S2-S5 appear is especially clear in figures 2.13 and 2.14.

2.3.4 Helical Travelling Waves

As noted earlier, Smith and Bodonyi (1982) hypothesised the existence of helical travelling waves in pipe flow. Landman (1990) attempted to calculate these without much success. More recently a study by Barkley and Tuckerman (2005) examined Couette flow with a computational domain close to the minimum required to sustain turbulence in two directions, and extended in the third (spanwise) direction. They a priori tilted the domain compared to the streamwise direction and so, in some sense, ‘forced’ the patterns which emerged to be slanted. These can be argued to be the Couette analogue of helical modes in pipe flow.
Figure 2.13: Friction factor against $Re_m$ for the known travelling waves, each for the value of $\alpha$ which minimises the lowest Reynolds number they reach. The numerals inside the plot give the corresponding rotational symmetry for each mode. The dashed line corresponds to laminar flow, while the dash-dot line is a parameterisation of experimental data. The inset plot is instead of the phase speed $C$ in units of the mean axial velocity against $Re_m$. 
Figure 2.14: The same plot as in figure 2.13, but this time the new travelling waves are shown at $\alpha = 0.75$. 
In attempt to calculate such modes in pipe flow we rewrote the travelling wave form as

\[ u(s, \phi, z, t) = u(s, \phi - \beta (z - ct) - \omega t, z - ct). \]  \hspace{1cm} (2.3.49)

This retains the usual linear link between \( z \) and \( t \), but introduces two new quantities. The helicity, \( \beta \), gives a linear link between \( \phi \) and \( z \) while \( \omega \) is a rotational phase speed necessary to reach finite levels of helicity. In the case of \( \beta = \omega = 0 \) we recover the usual travelling wave formulation.

For this case the derivative in both \( \beta \) and \( \omega \) is zero and so it should be possible to extend any travelling wave smoothly into the rotating, helical domain. (At this point let us make two points absolutely clear. Firstly it is the travelling wave that is rotating, not the pipe itself, and secondly the helical solutions do not represent a bifurcation from the other solutions. They are just an extension into another dimension of phase space.)

Within a helical framework it is not possible to support either \( S \)- or \( Z \)-symmetry, but \( \Omega \)-symmetry remains although modified to be

\[ \Omega_\beta : (u, v, w, p)(s, \phi, z) \rightarrow (u, v, w, p)(s, \phi + (1 - \beta/\alpha)\pi, z + \pi/\alpha). \]  \hspace{1cm} (2.3.50)

This reduction in symmetry leads to a significant increase in the quantity of storage required to be able to calculate these modes. Due to the extra symmetry of the mirror-symmetric travelling wave, we were able for the first time to resolve a helical mode properly.

The helical solutions look very similar to the travelling waves they have been extended from, except for a slight ‘twist’ to the shape (figure 2.15). Extending from the mirror-symmetric state, the helical modes bulge out of the \( c-Re \) plane connecting the upper and lower branches, and can achieve higher helicities (and rotational phase speeds) the further from the nose they are. When one extends from the asymmetric mode
Figure 2.15: Here are two cross sections of the helical flow extended from the mirror-symmetric solution at $Re = 1344$. They are from two slices $25D$ apart, and the helical rotation is immediately apparent. The parameters associated with it are $(\alpha, \beta, \omega) = (0.75, 0.019, -0.0011)$.

then it again bulges out a short way before reattaching to the helical surface extended from the mirror-symmetric state. For a given $\alpha$ we found that the helical modes repierced the $\beta = 0$ plane for a range of Reynolds numbers (in the case of $\alpha = 0.75$, this happened for $1165 < Re < 1330$). This gives a closed isola in the $c-\Omega$ plane which corresponds to rotating, non-helical travelling waves. A similar isola occurs of helical, non-rotating waves. A schematic diagram of the phase space is given in figure 2.16.

In all the cases examined, neither $\omega$ or $\beta$ ever exceed $O(10^{-2})$. This is well below the $O(1)$ results of Smith and Bodonyi (1982) and Barkley and Tuckerman (2005) and suggests that the flow has a preference for axially aligned streaks and rolls.

2.3.5 Relevance to Transition

The travelling wave that we set out to find, $S1$, has remarkable similarity to the time-averaged calculations of Schneider et al.. This is underlined by a calculation performed by Meseguer & Mellibovsky (2007). They initially calculated an edge trajectory in pipe flow. They searched this
Figure 2.16: This is a schematic drawing of phase space at $\alpha = 0.75$. The main hairpin shape in the $\beta = 0$ plane is the mirror symmetric solution, and it is from this the asymmetric mode bifurcates. Helical branches extend out of this plane connecting the upper and lower branches. The isola of rotating, non-helical travelling waves is shown by a dash-dot line. The dashed lines are inferred behaviour.
for a period of relatively stationary flow. Having identified such a section they time-averaged it to recover the state of Schneider & Eckhardt. From this averaged state they then extracted the roll structure and used precisely that as a forcing function to calculate a new travelling wave. This returned S1, perhaps unsurprisingly, but is confirmation that this mode really does seem to be central to the dynamics. A more progressive study was performed by Duguet et al. (2008b). In this they examined various quantities associated with the flow for periodic behaviour. At points that appeared to be close to periodic, they used the flow configuration as a starting guess for convergence onto an exact periodic solution. Invariably, the solution that was converged onto was the asymmetric travelling with one exception when it converged onto a slowly rotating version of the asymmetric wave. This study illustrated the key importance of particularly the asymmetric mode to transition in pipe flow.
Chapter 3

Highly Symmetric Travelling Waves

3.1 Introduction

3.1.1 A New Approach to Isolating Travelling Waves

In the previous chapter we discussed the relevance of travelling waves to turbulence in terms of the frequency with which they were visited within simulations. There was no a priori reason to believe the travelling waves would necessarily be visited - their validation as relevant was only suggested at by Hof et al. (2004) and not more confidently demonstrated until Kerswell and Tutty (2007) three years later.

It would be preferable to construct the travelling waves in such a way that they were of de facto relevance to observations. An ideal scenario would be to be able to pluck travelling waves directly from a cloud of turbulence as they were visited. In the paper Duguet et al. (2008b), exactly this was attempted.

The method they employed was to examine a turbulent signal for moments that might be close to a travelling wave. A defining characteristic
of all known travelling waves is that they are periodic (both spatially and temporally). If the flow is close to a travelling wave it can be assumed the flow signal of the flow should be close to periodic. Such points were identified either by visual inspection of velocity magnitudes at fixed points in the pipe, or by defining a scalar residual function which has to be minimised at each point in time.

The initial difficulty they ran into is the pipe turbulence appears to be too energetic for this approach. Any travelling waves are visited too fleetingly for a periodic signal to be detected. In order to circumvent this difficulty, Duguet et al. restricted their search to the laminar/turbulent boundary. As discussed in the previous chapter, this is populated by the same travelling waves, but this time their lower branches. Again, these provide the skeleton for the chaotic signals observed. However, here the dynamics are by definition less energetic and in a sense more slowly evolving.

Using machinery described in more detail in the following section they were able to examine the ‘saddle structure of the laminar/turbulent boundary’ in a 5D pipe at \( Re_m = 2875 \). They found numerous points within the flow which provided sufficiently good starting guesses to converge onto underlying travelling waves. In every instance the converged solution turned out to be the asymmetric travelling wave, S1, of the previous chapter or, on one occasion, a weakly rotating version of it.

In order to isolate new solutions, it was necessary for them to restrict their calculations to an area of phase space which excluded S1. This is most naturally achieved in one of two ways. One can reduce the length of the pipe so far that S1 no longer fits into it. At above critical Reynolds number, however, this is so short that one needs a very high \( Re \) in order to have sustained turbulence. Alternatively one can increase the periodicity in the azimuthal direction and work in the \( R_2 \)-symmetric subspace. This was the approach they adopted.
3.1.2 The Travelling Waves of Duguet et al.

Within this framework they identified three new travelling waves, which they referred to as ‘A3’, ‘C3’ and ‘D2’. That they were observed only in this subspace is due to the reduced number of unstable directions that they possess - any unstable direction that is not $R_2$-symmetric is automatically removed from the dynamics.

Although only the rotational symmetry was imposed on the flow, all three travelling waves further have $Z$-symmetry - they are mirror symmetric. A3 and C3 also exhibit $S$-symmetry (and consequently $\Omega$-symmetry). D2 is not $S$-symmetric, but is close to being in the sense of the simple indicator

$$\int_0^{2\pi} \int_0^1 (u - Su)^2 |_{z=0} s \, ds \, d\phi \quad \text{and} \quad \int_0^{2\pi} \int_0^1 (u + Su)^2 |_{z=0} s \, ds \, d\phi = O(10^{-3}). \quad (3.1.1)$$

That D2 is only $Z$-symmetric is noteworthy. Although there is no reason to expect the flow to prefer one symmetry over any another, previously all travelling waves in any on the canonical shear flows exhibit $S$-symmetry.

We took A3 and C3 and converted the numerical expression of them into the form described in Chapter 2. Putting this directly into the travelling wave codes, we were easily able to converge onto these new solutions. This did not prove to be the case with D2, we believe due to insufficient axial resolution. In an attempt to recover it we took a streamwise average of the roll structure and used this as a forcing function for the procedure described in 2.1.4, looking for $Z$-symmetric instabilities. This approach led to the discovery of a new travelling wave we will call Z2, as it is $Z$-symmetric and has rotational symmetry of two. It cannot be the same travelling wave, however, as it does not exist below a Reynolds number of 3250, while D2 was found at $Re = 2875$.

The travelling waves are strikingly different to those seen before as shown in figure 3.1 where they are compared to S2. Although they
possess the seemingly universal features of exact coherent structures in wall-bounded shear flows of wavy streaks and staggered quasi-streamwise rolls, they all also possess new features. A3 seems to concentrate both fast and slow streaks near the wall while the centre of the pipe is relatively untouched. C3 seems to have rolls operating in pairs to sustain the streaky structure. Both seem to hint at an almost double layered structure in the radial direction.

3.2 Two new classes of Travelling Waves

3.2.1 A3 and C3

Having successfully converged both A3 and C3 within the travelling wave code we were able to quickly and efficiently explore both of their positions in phase space. Both are born through the usual saddle node bifurcations at $(Re_m, \alpha) = (1125, 2.0)$ for A3 and $(1140, 1.2)$ for C3. These Reynolds numbers immediately stand out as being surprisingly low - lower than any of the travelling waves calculated in Wedin and Kerswell (2004), and perhaps most notably much lower than $Re_m = 1358$, the value at which the only other known two-fold rotationally symmetric travelling wave, S2, exists. Existence at low Reynolds number is not required to allow relevance to observed dynamics - only that it exists at sub-transitional Reynolds number. However the calculations by Kerswell and Tutty (2007) and others do suggest a correlation between travelling waves spawned at low Reynolds numbers and those which are observed in turbulence. This coupled with their unusual appearance and the fact that edge trajectories are known to pass near them is enough to suggest that these travelling waves are worth investigating.

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Figure 3.1: The travelling waves $A_3$ (top left), $C_3$ (top right) and $Z_2$ (bottom left). $S_2$ (bottom right) is also included for comparison purposes. All are shown at their respective values of $\alpha^*$ (see table 1) and at $Re = 2400$ except for $Z_2$ which is shown at $Re = 3860$. The contours indicate the magnitude of the streamwise velocity difference from the underlying laminar flow. The contours and colouring is standardised across the plots (and more generally figures 3.2 and 3.3) with contours increments of $0.02U$ running from $-0.19U$ (dark red) to $0.19U$ (white) (the colour outside of the pipes indicates zero). Arrows indicate the cross-stream velocities.
3.2.2 Generalisation of Rotational Symmetry

There is nothing special about the two-fold rotationally symmetric subspace. Just as S2 appears to be related to S3, S4, etc, we can expect there to be more travelling waves related to A3 and C3. The question is how can we find these higher (rotational) symmetry counterparts. One method would to repeat the calculations of Duguet et al. but within a $\mathbb{R}_n$-pipe, for $n = 3, 4, 5,\ldots$. This calculation, however, is time consuming and has a random, unrepeatable element to it.

Instead we return to the method of section 2.1.4. Here we briefly outlined the way in which the travelling wave we have classified as U1 was found. The importance of this travelling wave was not highlighted at the time, but this in fact is the very first known travelling wave which exhibits both $S-$ and $Z-$symmetry. It also demonstrates the feasibility of finding these travelling waves with this method.

The one modification we make to the approach comes from the observation that both A3 and C3 appear to have, to a greater or lesser extent, two layers of streaks. In the original method we denoted all the least decaying eigenvalues as $\lambda_{m_0 i}$, with $m_0$ being the rotational symmetry and $i$ being the number of zeros of the function in the range $0 < s \leq 1$. This motivated us to use $i > 1$ unlike previously.

This methodology worked remarkably well and led to a number of new travelling waves with little work. These new waves made it clear that we had two separate classes of travelling waves, which we will denote as the M and N classes. Each class is made up by a separate family of travelling waves for rotational symmetry, $m_0$. We introduce the nomenclature $Mm_0$ and $Nm_0$ for the $R_{m_0}$ member of the M and N classes respectively. A3 belongs to the M2 family, while C3 corresponds to N2. In this way we were able to uncover M2, M4, M5 and M6 of the M class and N2, N3, N4 and N5. The missing elements were M1, M3 and N1.

We were able to identify N1 and M3 by continuation in rotational
symmetry. This is performed using precisely the same branch following techniques as before, but this time we varied $m_0$. While for non-integer values of $m_0$ the ‘solutions’ have no physical meaning, once a new integer value is reached, physicality returns. The only subtlety required is that one must begin from an odd (even) value of $m_0$ if one wishes to find a new travelling wave with odd (even) rotational symmetry due to the parities implied by equations 2.1.31-2.1.33. Inspection of the physical appearances of the new found travelling waves revealed that M1 was already known - it is the mirror-symmetric travelling wave of the previous section.

The physical appearances of the travelling waves are shown in figure 3.2 (the M-class) and figure 3.3 (the N-class). The non-rotationally symmetric travelling waves are shown separately in figure 3.4 along with U1 for comparison.

3.2.3 Phase Space

Having identified two new families of travelling waves the obvious next step was to explore their position in phase space to see where they fit into the wider picture. Thus for each member of the two families we identified the value of $\alpha = \alpha^*$ which enables the travelling wave to reach the lowest possible value of $Re$, precisely as in section 1.3.3. The results are summarised in table 3.1.

The bifurcation points for the M-class of travelling waves monotonically increase in both $Re$ and $\alpha$ with the rotational symmetry. This is not strictly true for the N-class, where although the value of $\alpha^*$ increase monotonically with $m_0$, $Re$ does not. N3 exists at the lowest value of $Re$ with the travelling waves on either side of it appearing at progressively higher Reynolds numbers (this is curiously reminiscent of the S-class travelling waves originally discovered). Both classes exist, however, at markedly low Reynolds numbers. The full exploration of phase space is shown in figures 3.5 and 3.6. These plots nicely illustrate
Figure 3.2: Slices of the lower branch M-class travelling solutions at $Re = 2400$ and at each travelling wave’s respective value of $\alpha^*$. M2 (A3, top left), M3 (top right), M4 (bottom left) and M5 (bottom right). Contour levels of the streamwise velocity perturbation are in increments of $0.02U$ running from $-0.19U$ (dark red) to $0.19U$ (white).
Figure 3.3: Slices of the lower branch N-class travelling wave solutions at $Re = 2400$ and at each travelling wave's respective value of $\alpha^*$. N2 (C3, top left), N3 (top right), N4 (bottom left) and N5 (bottom right). Contour levels of the streamwise velocity perturbation are in increments of 0.02$U$ running from $-0.19U$(dark red) to 0.19$U$(white).
Figure 3.4: Slices of the lower-branch, non-rotationally-symmetric traveling wave solutions M1 (left), N1 (middle) and U1 (right). The first two are at $Re = 2400$, while the last is at $Re = 3046$, and each traveling wave is shown at its respective value of $\alpha^*$. Contour levels of the streamwise velocity perturbation are in increments of $0.02U$ running from $-0.19U$ (dark red) to $0.19U$ (white).

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<th>Travelling Wave</th>
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<td>$Z &amp; R_2$</td>
<td>$&lt; 2875$</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 3.1: Summary of all the travelling waves introduced in this chapter (including the re-named mirror-symmetric travelling wave).
Figure 3.5: A plot of the phase speed $C$ (in units of $U$) against the axial wavenumber $\alpha$ (in units of $2/D$). The contours correspond to different $Re$ with the loops (M-class/N-class travelling waves shown using blue solid/red dashed lines) constricting as they move towards the saddle nose indicating $\alpha^*$. The various $Re$ for each travelling wave are as follows: M1 776 & 820, M2 1125 & 1145, M3 1552 & 1650, M4 1846 & 2037, M5 2148 & 2400, M6 2540 & 2662; N1 1581 & 1747, N2 1150 & 1318, N3 1038, 1050 & 1120, N4 1292 & 1300, N5 1629 & 1652.

the order in which the travelling waves emerge.

Special mention is given to the plot of friction factor, figure 3.7. This starkly illustrates a number of fundamental points associated with these travelling waves. Firstly it is readily apparent on this plot by just how much these new travelling waves precede their predecessors in terms of how low in $Re$ they appear. A slight concern about explaining turbulence in terms of visiting numerous travelling waves is seen in the values of the friction factor that the travelling waves take. The upper branch solutions of the original travelling wave are a reasonable match for the experimentally observed average friction factor of turbulence. The key point here is in the use of the word ‘average’. The neat scaling of this
Figure 3.6: A plot of $C$ against $Re$ for $\alpha^*$ (see 3.1) that gives the lowest saddle node bifurcation for each travelling wave (M-class/N-class waves shown using blue solid/red dashed lines). The non-mirror symmetric S-class of Faisst & Eckhardt (2003) and Wedin & Kerswell (2004) which generally appear at higher $Re$ are shown using black dotted lines. Typical resolutions used (in the nomenclature of Wedin & Kerswell (2004)) are: M1 (14,25,5), M2 (12,30,6), M3 (10,30,10), M4 (10,30,8), M5 & M6 (10,35,8); N1 (14,30,5), N2 (12,30,6), N3 & N4 (10,30,8), N5 (8,35,8).
Figure 3.7: The friction factor, $\Lambda$, against $Re$ for the newly found travelling waves, as well as the asymmetric wave (M1). The lower dotted line corresponds to the laminar state, and the upper dotted line to the log-law parameterisation of experimental data, $1/\sqrt{\Lambda} = 2.0 \log(Re_m\sqrt{\Lambda}) - 0.8$ (see Schlichting 1968, equation (20.30)). The inset shows the same plot but for the travelling waves S2–S6 (only selected curves drawn for clarity) and Z2 which appears at much higher $Re$. The earlier onset in $Re$, and significantly higher friction factors of the new $M$ and $N$-class travelling waves is clearly apparent.

The averaged line belies the important point that the instantaneous friction factors can vary greatly on either side of it. Thus the originally found travelling waves do not appear to be able to explain a lot of the observed dynamics when a much higher level of ‘non-laminarity’ is observed. The two new families of travelling waves (especially the N-class) achieve much higher levels of friction factor than those that went before.

### 3.2.4 Stability

As we have been at pains to make clear, other than the laminar state, all known exact solutions in pipe flow are linearly unstable. The fact that no coherent structure is ever observed experimentally leads us to
conclude that in fact there are indeed no linearly stable non-trivial solutions. Stability analysis can tell one a great deal about a system from a dynamical systems point of view. Perhaps the most intriguing possibility that arises from it is the possibility of the number of unstable directions changing at a particular Reynolds number - the real part of the eigenvalue crossing from the positive to the negative half plane. This is a bifurcation point, and represents the seed of a secondary solution splitting off the travelling wave. In the generic case (treated in the next chapter), the imaginary part of the eigenvalue will be non-zero (in this scenario there must in fact be two eigenvalues making the change from stable to unstable - conjugates of each other). This corresponds to the perturbation having a frequency relative to the underlying travelling wave and a more complicated structure arises. If the imaginary part is zero then the relative frequency is also zero, and so it is another travelling wave. This was the way in which the asymmetric travelling wave appears, bifurcating from what we now call M1. Attempts to calculate a closed loop in \((\alpha, c)\) space of S3 at \(Re = 2400\) led to figure 3.8. This plot is remarkably illustrative. Firstly we clearly see S3 has bifurcated off N3. This comes as something of a shock, as it had been thought that all of the original travelling waves were isolated in phase space. It also shows just how much more enveloping of phase space N3 is than S3, and how much smoother it appears in phase space.

We now know of two travelling waves which bifurcate off other waves - S1 from M1 and S3 from N3. This undoubtedly suggests that there are more bifurcations to be found and a more exhaustive stability analysis of N2 was performed. The number of unstable directions a travelling wave possesses depends upon the space in which one performs the stability analysis. Theoretically one should consider an infinitely long pipe with the wave repeating eternally along it and then calculate the stability of this flow. This is clearly not possible. In order to restrict ourselves to a more manageable problem we consider only perturbations to the
Figure 3.8: The S3-N3 family connection at $Re = 2400$ on a friction factor $\Lambda$ versus $\alpha$ plot. This plot for S3 appeared in figure 1 of Kerswell & Tutty (2007). Now the bifurcation from the more symmetric N3 family is clear. The resolution is (8,30,6) in the nomenclature of Wedin & Kerswell (2004) throughout (the N3 curve is essentially unchanged for the higher resolution of (10,30,8)). The lower dotted line represents the lower bound given by the Hagen-Poiseuille solution ($\Lambda_{lam} = 64/Re$) and the upper dashed line corresponds to the $Re = 2400$ value of the log-law parameterisation of the experimental data $1/\sqrt{\Lambda} = 2.0 \log(Re\sqrt{\Lambda}) - 0.8$ (see Schlichting 1968, equation (20.30)).
travelling wave which share both its axial wavelength and rotational symmetry. We wish to consider both the upper and lower branches of the travelling wave. This presents us with a problem as far as plotting the results goes. We cannot plot growth rate against Reynolds number as even for just one value of $\alpha$ the wave is multivalued in Re. Another look at figure 3.6 shows that $C$ parameterises both branches uniquely. In figure 3.9 we therefore plot growth rate against $C$. In this figure, every time a solid line crosses the neutral stability axis a new travelling wave is born (except at the saddle node). Four new travelling waves are thus created with varying symmetries, and a fifth appears imminent further along the upper branch.

### 3.3 Implications

The new travelling waves discussed in this chapter seem to be more fundamental than those previously found, and seem to structure phase space. They are the first travelling waves that are known to appear (in terms of Reynolds number), and it seems that the S-class of travelling wave come in as secondary travelling waves bifurcating off their more symmetric counterparts. The stability analysis of N2 in particular suggests that there are many more travelling waves than originally appreciated and that the original work had in essence just touched on the boundaries of what exists.

Prior to this work, travelling waves had been viewed as being isolated points in phase space of relatively restricted form. The rich bifurcation structure of these travelling waves demonstrates that there is much more interplay between the travelling waves than might have otherwise been expected. As already emphasised, the original work only succeeded in isolating travelling waves of shift-&-reflect symmetry. The solutions here possess this symmetry but also two further symmetries. Via bifurcations they then give birth to travelling waves which can be
Figure 3.9: Stability of N\textsubscript{2} within the \( \mathbb{R}_2 \)-symmetric subspace against the phase speed \( C \) as \( Re \) increases away from the saddle node bifurcation at 1141 (\( \alpha = \alpha^* \) and the corresponding \( Re \) is across the upper x-axis). Each line either indicates the locus of a real eigenvalue (solid) or a complex conjugate pair (dashed) as \( Re \) changes. The blue (dark) lines correspond to those which are symmetric under \( \mathcal{S} \) while the green (light) lines are anti-symmetric under \( \mathcal{S} \). The vertical dotted line indicates the saddle node, of which the inset is a close up. There are bifurcations at \( C = 1.33, 1.38, 1.385, 1.388, 1.389 \) (the saddle node), 1.44, 1.47 and 1.50. The dotted green indicates a stable complex conjugate pair which was difficult to resolve. The inset plot provides a closeup of the the saddle node.
$Z-$ or $\Omega-$symmetric. The richness of and complexity of phase space has been highlighted with these new modes.

The new travelling waves also cover a wider region of phase space, reaching more energetic, and higher friction factor states than before. Previously travelling waves only appeared to be in the regions of phase space corresponding to relatively low energy turbulence. The new solutions now extend into the upper reaches of saturated turbulence. The greater variety in physical appearance seems to emphasise this. If turbulence can take on radically different appearances at different instances in time, then in order to formulate a model of turbulence in terms of exact solutions we need a variety of forms of coherent structures.

We end by noting that Willis and Kerswell (2008) suggested that more energetic travelling waves were required in order to describe turbulence. These travelling waves certainly fulfil that requirement. It would be naïve to think that we have now provided a complete catalogue of travelling waves or even of the most important ones, however. There are likely many more solutions out there to be found, and perhaps some of even more central importance.
The work presented in the following chapter was done in collaboration with Dr. Yohann Duguet, and was first published in Duguet et al. (2008a). To perform the calculations presented here two codes were required. The travelling wave code from the previous chapters was used to pursue the travelling waves required and perform the necessary stability analysis. The relative periodic orbit was calculated using the code of Dr. Duguet.
Chapter 4

Relative Periodic Orbits in Transitional Pipe Flow

4.1 Abstract

A dynamical system description of the transition process in shear flows with no linear instability starts with a knowledge of exact coherent solutions, among them travelling waves (TWs) and relative periodic orbits (RPOs). We describe a numerical method to find such solutions in pipe flow and apply it in the vicinity of a Hopf bifurcation from a TW which looks to be especially relevant for transition. The dominant structural feature of the RPO solution is the presence of weakly modulated streaks. This RPO, like the TW from which it bifurcates, sits on the laminar-turbulent boundary separating initial conditions which lead to turbulence from those which immediately relaminarise.

4.2 Introduction

Ever since the experiments of Osborne Reynolds in 1883 (Reynolds, 1883), it has been known that the flow of a Newtonian fluid in a cir-
circular pipe undergoes an abrupt transition from a laminar to a turbulent state. This transition is governed by one dimensionless parameter, the Reynolds number $Re := UD/\nu$ ($U$ is the bulk velocity, $D$ the pipe diameter and $\nu$ the kinematic viscosity of the fluid). Transition is usually observed experimentally for $Re \sim 2000 \pm 250$ but can be delayed to larger values in especially well-controlled experiments (Darbyshire and Mullin, 1995; Pfenniger, 1961). The laminar state (Hagen-Poiseuille flow) is believed stable for all $Re$, so that transition to turbulence cannot be explained by an ordered sequence of bifurcations starting from the basic state. Instead, pipe flow belongs to a more general class of parallel shear flows, including plane Couette and plane Poiseuille flow, which undergo finite-amplitude instability (Drazin and Reid, 1985). The transitional dynamics of these flows then appears organised around the presence of other solutions disconnected from the laminar state. These new solutions are expected to have a relatively simple structure compared to the turbulent dynamics and to be unstable (Waleffe, 1998; Kerswell, 2005; Gibson et al., 2008). These expectations are conceptually satisfying for two reasons. Firstly, relatively simple three-dimensional mechanisms (e.g. the ‘self-sustaining process’) have been identified in low-$Re$ flows (Hamilton et al., 1995; Waleffe, 2001) allowing turbulence to be sustained over long times, which evokes the possibility of organised dynamics. Secondly, instability is needed to rationalise why neither steady nor periodic exact states have been observed in any three-dimensional turbulent flow so far.

The fact that statistically recurrent states have been clearly observed, e.g. streaky velocity fields with an intrinsic wavelength (Hamilton et al., 1995) in most wall-bounded shear flows, is most easily explained by the existence of interconnected (unstable) exactly recurrent solutions of the governing equations.

Progress in numerical computation has first led to the discovery of some exact solutions in plane shear flows (Nagata, 1990; Waleffe, 1998, 2001) and more recently pipe flow (Faisst and Eckhardt, 2003; Wedin
and Kerswell, 2004). They are all equilibria (steady states) or relative equilibria (travelling waves which are steady in a moving frame) and all possess a small number of unstable eigendirections. They appear through saddle-node bifurcations and are disconnected from the laminar state. There is a tremendous interest in these solutions as very similar structures have been observed transiently in experiments (Hof et al., 2004; Hof et al., 2005). It is worth emphasizing that their simple dynamics (steadiness or constant speed propagation) is inherent to the method used to seek them, and undoubtedly hides a larger variety of other solutions. At some point the physical description of the flow in terms of three-dimensional structures ceases to be enlightening and it is useful to think in terms of a trajectory in phase space. Formally, the Navier-Stokes equations can be projected onto any infinite-dimensional basis of incompressible velocity fields to yield an autonomous dynamical system of the form $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}, Re)$. The choice of the basis, and hence the particular projection, is not crucial. The topology of the phase space associated to pipe flow near transition is thought to be organised around one locally attractive point (the 'laminar' flow) and a chaotic saddle, i.e. a set of unstable solutions and their heteroclinic and homoclinic connections (Mullin and Kerswell, 2004; Faisst and Eckhardt, 2003). From this point of view, it is necessary to know which states the saddle is made of in order to understand the dynamics near transition. It has been demonstrated that the unstable solutions known so far in pipe flow, the travelling waves (TWs), are visited only for about $\mathcal{O}(10-20\%)$ of the time (Kerswell and Tutty, 2007; Schneider et al., 2007a). This indicates that, provided the picture is correct, more solutions of a different type have to be sought. The next level up in the hierarchy of solutions from relative equilibria is relative periodic orbits (RPOs). They are an extension of periodic orbits (in the same way as travelling waves correspond to degenerate fixed points) due to the invariance of the equations with respect to the azimuthal and axial directions. Physically, a RPO
corresponds to a flow which repeats itself periodically in a given moving frame, translating and/or rotating along the pipe axis at a constant rate. In the cylindrical coordinate system \((s, \theta, z)\) aligned with the pipe axis, it is a time-dependent velocity field \(v(s, \theta, z, t)\) satisfying

\[
v(s, \theta, z, t + T) = v(s, \theta + \Delta \theta, z + \Delta z, t)
\]

for some constants \(T\), \(\Delta \theta\) and \(\Delta z\). The initial motivation behind the search for periodic behaviour is Periodic Orbit Theory (Christiansen et al., 1997), which states that any dynamical average of a smooth chaotic dynamical system can be evaluated in a deterministic way by finding all unstable periodic orbits of the system up to a certain period, and carefully averaging over them. Consideration of continuous symmetries in the original PDEs - here translation in \(z\) and rotation in \(\theta\) - extends this expansion over all relative periodic orbits (Lopez et al., 2005). Even if complete knowledge of all periodic orbits of the system seems computationally ambitious today, a start needs to be made in developing numerical tools to find them in anticipation of greater computing power tomorrow. Some progress has already been made in plane Couette flow with the recent discovery of periodic (Kawahara and Kida, 2001) and relative periodic orbits (Viswanath, 2007) embedded in turbulence, albeit in small flow domains. Beyond the obvious computational savings, restricting the largest wavelength in the flow has proved a very useful tool for isolating key dynamics (Hamilton et al., 1995). In the same spirit, we will here concentrate on relatively short ‘periodic’ pipes only a few diameters long.

In this study we are interested in RPOs which are located in the laminar-turbulent boundary, the invariant subset of phase-space on which trajectories neither relaminarise nor become turbulent. States belonging to this subset can be visited transiently by phase-space trajectories during the transition process. Phase-space trajectories constrained to
remain precisely on this laminar-turbulent boundary have been found to approach a chaotic attractor (Schneider et al., 2007b) centred on one particular TW solution, the ‘asymmetric TW’ found by Pringle and Ker-swell (2007). The importance of this particular TW has been confirmed by further calculations which show that trajectories constrained to lie on this laminar-turbulent boundary recurrently approach this TW Duguet et al. (2008b). We therefore concentrate on looking for RPOs which bifurcate from this asymmetric TW.

The paper is organised as follows. Section 4.3 explains in detail the numerical method chosen to find RPOs from a starting guess and section 4.4 describes the new branch of RPOs found. Subsection 4.5 confirms that the RPO is on the laminar-turbulent boundary and then explores the likelihood of connections between the RPO and other TWs. Section 4.6 discusses the relevance of the RPO to the transition process as well as the numerical limitations of the method.

4.3 Numerical Procedure

4.3.1 Governing equations

We consider the incompressible flow of Newtonian fluid in a cylindrical pipe and adopt the usual set of cylindrical coordinates \((s, \theta, z)\) and velocity components \(\mathbf{u} = u\hat{s} + v\hat{\theta} + w\hat{z}\). Units of length and flow speed are taken as the pipe radius and the bulk speed \(U\) so that the computational domain is \(s, \theta, z \in [0, 1] \times [0, 2\pi] \times [0, L]\), where \(L = 2\pi/\alpha\) is the length of the pipe. The non-dimensionalised incompressible three-dimensional Navier-Stokes equations read

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u},
\]

\[
\nabla \cdot \mathbf{u} = 0,
\]

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where recall \( \text{Re} := UD/\nu \) and the flow is driven by a constant mass-flux condition

\[
\frac{1}{\pi} \int_0^1 \int_0^{2\pi} w \, s \, ds \, d\theta = 1,
\]

(4.3.4)
as in recent experiments (Darbyshire and Mullin, 1995; Peixinho and Mullin, 2006, 2007). Although time is calculated in units of \( D/2U \), all times are quoted hereafter in usual units of \( D/U \). The boundary conditions are periodicity across the pipe length \( u(s, \theta, z) = u(s, \theta, z + L) \) and no-slip on the walls \( u(1, \theta, z) = 0 \). In the non-dimensionalisation used, the expression of the Hagen-Poiseuille flow is \( u_{HP} = 2(1 - s^2)\hat{z} \).

### 4.3.2 Time-Stepping Code

The basic tool for the numerical determination of periodic orbits is an accurate time-stepping code. The direct numerical simulation code used in references (Willis and Kerswell, 2007; Willis and Kerswell, 2008, 2009) was adopted which is based upon the toroidal-poloidal representation of the velocity field (Marqués, 1990).

\[
u = \nabla \times (\Psi \hat{z}) + \nabla \times \nabla \times (\Phi \hat{z}).
\]

(4.3.5)
The two scalar potentials \( \Phi \) and \( \Psi \) are discretised using high-order finite differences in the radial direction \( s \), and with spectral Fourier expansions in the azimuthal \( \theta \) and axial \( z \) directions. At any discrete radial location \( s_j \), \( (j = 1, ..., N) \), each of the potentials (e.g. here \( \Phi \)) is therefore discretised according to the formula:

\[
\Phi(s_j, \theta, z, t; \alpha) = \sum_{k=-K}^{K} \sum_{m=-M}^{M} \Phi_{jk}(t)e^{i(m\theta + \alpha kz)}
\]

(4.3.6)
The resolution of a given calculation is described by a triplet \( (N, M, K) \). The phase-space associated to the Navier-Stokes equations is the set
of complex coefficients \( \{ \Phi_{jkm}, \Psi_{jkm} \} \), which corresponds to a dynamical system with \( n \sim 8MNK \) real degrees of freedom (typically \( n = \mathcal{O}(10^5) \)). Its metric is defined by the Euclidean scalar product \( \langle , \rangle \). A shift back in physical space by \((\Delta z, \Delta \theta)\) corresponds in phase-space to the transformation:

\[
(\Psi_{jkm}, \Phi_{jkm}) \rightarrow (\Psi_{jkm}, \Phi_{jkm}) e^{-i(m\Delta \theta + \alpha k \Delta z)}
\] (4.3.7)

Time discretisation is of second-order, using a Crank-Nicholson scheme for the diffusion term and an Euler predictor step for the non-linear terms. In this study we confine our calculations to a pipe of length \( L = 2\pi/0.75 \sim 8.37 \) radii. This length has been chosen because it is a wavelength for which the asymmetric TW (Pringle and Kerswell, 2007) is on the laminar-turbulent boundary and for which most data were already available.

### 4.3.3 The Newton-Krylov Method

**Periodic Orbits**

A periodic orbit (of period \( T \)) of a dynamical system \( \dot{X} = F(X) \) is sought as a solution \((X, T)\) of the equation \( G(X) = 0 \), where

\[
G(X) := \phi^T(X) - X.
\] (4.3.8)

Here \( \phi^T(X) \) refers to the point at time \( t = T \) on the trajectory starting from \( X \) at time \( t = 0 \). A value of the period \( T \) has to be chosen to uniquely define \( G(X) \). We choose \( T \) as the time minimising the \( X \)-dependent scalar function \( g : t \rightarrow |\phi^t(X) - X|^2 \) over a given time interval. This is done by accurate interpolation of the zeros of its derivative, the scalar function \( g' : t \rightarrow 2(\phi^t(X) - X, \partial \phi^t(X)/\partial t) \). Figure 4.1 shows schematically that the time \( t = T \) is picked up along the trajectory when its tangent vector is orthogonal to the difference vector \( \phi^t(X) - X \),
which ensures that the trajectory has come back closest to the initial point. A standard method to search for zeros of $G$ in low-dimensional systems (typically $n \lesssim \mathcal{O}(10^3)$) is the Newton-Raphson method. This iterative algorithm, based on successive Taylor expansions of $G$, produces a sequence of iterates $X^{(k)}$, $k \geq 0$ defined by

$$X^{(k+1)} = X^{(k)} + \delta X^{(k)}, \quad (4.3.9)$$

where

$$J(X^{(k)})\delta X^{(k)} = -G(X^{(k)}), \quad (4.3.10)$$

$J$ being the Jacobian matrix associated to $G$. When no analytical expression for $J$ is available, it is generally computed by finite differences, at the cost of one evaluation of $G$ for each column of the matrix. In very high dimension $n \gtrsim \mathcal{O}(10^4)$, however, the time needed for $n + 1$
evaluations of $G$, as well as the storage of $J$ (not to mention solving equation (4.3.10)), can become prohibitive, thus we turn our attention towards matrix-free Inexact Newton-Krylov methods.

At every Newton step $k$, we solve the system (4.3.10) using the GMRES algorithm (Saad and Schultz, 1986). This iterative linear solver only requires matrix-vector products that can be evaluated using finite differences, since for a vector $p$ and $\epsilon$ sufficiently small (e.g. $10^{-7}$) we have:

$$Jp = \frac{G(X^{(k)} + \epsilon p) - G(X^{(k)})}{\epsilon} + O(\epsilon). \quad (4.3.11)$$

Hence no matrix needs to be stored explicitly. Using (4.3.11), GMRES iteratively builds an orthogonal basis of the Krylov subspace spanned by $\{r_0, Jr_0, J^2r_0, \ldots\}$, starting from $r_0 = -G(X^{(k)})$. At each GMRES step, a Gram-Schmidt procedure reorthonormalises the Krylov subspace producing a basis $\{V_0, V_1, V_2, \ldots\}$. Then an approximation to the solution of (4.3.10) is constructed on this basis, until convergence is decided according to the criterion:

$$|J^{(k)}\delta X^{(k)} + G^{(k)}| \leq \eta^{(k)}|G^{(k)}|. \quad (4.3.12)$$

The ‘forcing’ constant $\eta^{(k)}$ appearing in formula (4.3.12) is ideally zero, and numerically it should be in principle very small. However, non-vanishing values of $\eta$ can allow for a much quicker convergence of the overall Newton scheme, by avoiding useless oversolving. Values of $\eta \sim O(10^{-1})$ have produced faster convergence and even better performances were observed when updating $\eta^{(k)}$ with the choice 2 of Eisenstat and Walker (1996).
Double Dogleg Step

Newton-Raphson algorithms are known to converge only if the initial guess is sufficiently close to a zero of the function $\mathbf{G}$, which in high dimension often means no convergence at all. Moreover, even close to a solution, Newton-Raphson steps either can be too large or nearly orthogonal to the gradient of $|\mathbf{G}|^2$, in which case the algorithm stagnates and the classical strategy of linear backtracking (also called ‘damped Newton’) is of no help (Tuminaro et al., 2002). To remedy this, it is useful to embed the algorithm into what is commonly called a ‘globally’ convergent strategy. Here ‘global’ does not mean that convergence is achieved whatever the starting point, but instead that the basin of attraction of a solution is reasonably enlarged. We adopt the ‘double dogleg step’ method proposed by Dennis and Schnabel (1996), which is a subclass of the ‘trust region’ algorithms (see Viswanath (2007) for an implementation of another subclass called the ‘hook step’ approach).

All these methods start from the knowledge of the Newton step $\delta X_N$ (here we omit the superscript $(k)$), and aim at constructing a new vector $\delta X$ whose norm is bounded by a ‘trust length’ $\delta_c$. The way $\delta X$ is chosen is described in the next paragraph. At the first stage of the loop, $\delta X = \delta X_N$ and $\delta_c = |\delta X_N|$. Then at every iteration of the trust region algorithm, a check is made to see whether $|\mathbf{G} (\mathbf{X}^{(k)} + \delta X)|$ is sufficiently smaller than $|\mathbf{G} (\mathbf{X}^{(k)})|$. If it is, then the loop is finished and we set $\mathbf{X}^{(k+1)}$ to $\mathbf{X}^{(k)} + \delta X$; if not, then $\delta_c$ is divided by 2, a new vector $\delta X$ is computed and the process is repeated.

In the case of the double dogleg step, two descent directions for $|\mathbf{G}|^2$ are used to generate the new vector $\delta X$: the first one is the Newton step $\delta X_N$, the second is the steepest descent direction (see figure 4.2):

$$\frac{1}{2} \nabla |\mathbf{G}|^2 = -\mathbf{J}^T \mathbf{G}, \quad (4.3.13)$$

where $\mathbf{J}^T$ is the transpose of the Jacobian matrix $\mathbf{J}$. We can define
the Cauchy Point $X^{(k)} + \delta X_{CP}$, where the vector $\delta X_{CP}$ minimizes the quadratic form $\delta X \to |J\delta X + G|^2$ along the steepest descent direction (note that the overall minimizer of this quadratic form defines the Newton step $\delta X_N$). Projection of $\delta X_{CP}$ onto the Krylov space (already built to find $\delta X_N$ at the same Newton iteration) allows a low-cost approximation of $\delta X_{CP}$, as long as the dimension of the Krylov subspace itself is low. If the $V_i$’s form an orthonormal basis of the Krylov subspace and $K$ is its dimension, we have:

$$\nabla |G|^2 \sim \sum_{i=1}^{K} \langle J^t G, V_i \rangle V_i = \sum_{i=1}^{K} \langle G, J V_i \rangle V_i.$$ (4.3.14)

In (4.3.14), the scalar coefficients $\langle G, J V_i \rangle$ are already known up for $1 \leq i \leq K$ since (normalized by $-|r_0|$) they form the first row of the Hessenberg matrix used for the GMRES inversion (Saad and Schultz, 1986).

Finally, given $\delta X_N$ and $\delta X_{CP}$, the double dogleg algorithm finds the intersection between the piecewise linear curve linking the Cauchy Point to $X + \mu \delta X_N$, and the ball of radius $\delta_c$. So the new trial vector is chosen as:

$$\delta X = \delta X_{CP} + \lambda_c (\mu \delta X_N - \delta X_{CP}).$$ (4.3.15)

where $\mu$ is a constant set to 0.8 (following Dennis and Schnabel (1996)) and $\lambda_c > 0$ is chosen such that $|\delta X| = \delta_c$. If $|\delta_c| \leq |\delta X_{CP}|$, the dogleg curve along which optimisation is achieved is simply the steepest-descent direction of $|G|^2$.

**Relative Periodic Orbits**

Allowing for shifts in $\theta$ and $z$ implies a modification of the function $\mathbf{G}$. If we seek a relative periodic orbit satisfying $\mathbf{v}(r, \theta, z, t + T) = \mathbf{v}(r, \theta + \Delta \theta, z + \Delta z, t)$ for any $t$, we have to change $\mathbf{G}(X)$ into $\mathbf{G}(X) =$
Figure 4.2: Schematic view of the double dogleg technique (adapted from Dennis and Schnabel (1996)) in phase space. \( X^{(k)} \) is the state at the \( k^{th} \) Newton iteration, \( \delta X_N \) represents the Newton step computed using the inexact GMRES method, and \(-\nabla|G|^2\) is the steepest descent direction of the residual from \( X^{(k)} \). At a given step, a new state \( X^{(k+1)} = X^{(k)} + \delta X \) is suggested as the intersection between the double dogleg curve (medium thick line) and the ball of radius \( \delta_c \).
In this expression, $\phi^T(X)$ is shifted back in the $\theta$- and $z$-directions using (4.3.7), before being compared to the initial point of the orbit. $T$ is still chosen as the time minimising $|G|^2$ at each new iteration, whereas $\Delta \theta$ and $\Delta z$ are considered as two extra variables of the Newton algorithm (Viswanath, 2007): we define a $n+2$-dimensional vector $X^+ = (X, \Delta \theta, \Delta z)$ and the function $G^+ : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ such that

$$G_i^+(X^+) = G_i(X), \quad (i = 1, \ldots, n),$$

while the two last components of $G^+$ are defined by the scalar products in $\mathbb{R}^n$:

$$G_{n+1}^+(X^+) = \left\langle G, \frac{\partial \phi^T}{\partial \Delta \theta} \right\rangle,$$

$$G_{n+2}^+(X^+) = \left\langle G, \frac{\partial \phi^T}{\partial \Delta z} \right\rangle.$$

These two derivatives are easily computed by central finite differences. Starting from a guess $X_0$ (which includes a starting point in phase space as well as two initial values of $\Delta \theta$ and $\Delta z$), the Newton-Krylov method is now applied to locate zeros of the function $G^+$. It’s worth remarking that although $T$ looks to be on the same footing as $\Delta \theta$ and $\Delta z$, this is not the case. The condition $\langle G, \partial \phi^T / \partial t \rangle = 0$ which defines $T$ is imposed at every Newton/dogleg iteration whereas the equivalent conditions for $\Delta \theta$ and $\Delta z$ are only truly reached when convergence to a RPO has been achieved.

Non-rotating TW solutions are a special case of RPOs, when $T$ and $\Delta z$ are linked by the relation $\Delta z = cT$, with $c$ the axial propagation speed of the wave. Because of this degeneracy, TW solutions with zero azimuthal speed can be sought, using the same algorithm, by fixing the
value of $\Delta z$ to $2\pi/\alpha$, and setting $G_{n+2} = 0$.

### 4.4 Results

#### 4.4.1 Hopf Bifurcation of Travelling Waves

The Newton-Krylov algorithm described in Section 4.3 was used to find a RPO bifurcating off the asymmetric TW branch in a periodic pipe of length $L \sim 4.18 \, D$ ($\alpha = 0.75$) where it is known to be embedded in the laminar-turbulent boundary. The asymmetric TW branch originates from a supercritical bifurcation at $Re \sim 1770$ off a “mirror-symmetric” branch of TWs (Pringle and Kerswell, 2007). The asymmetric TW propagates axially with a phase speed $c = 1.34 \, U$ but does not rotate. For $Re \sim 1785$, it displays a pair of high-speed streaks sandwiching an eccentric low-speed streak (see Figure 4.3). A streamwise vortex tube is present in the vicinity of the low-speed streak, ensuring the regeneration of the streak via a lift-up effect. The two high-speed regions extend to the other side of the cross-section, and another low-speed streak of lesser intensity is present on the right of the original one. As $Re$ increases, the solution becomes more asymmetric and the streaks on the right of the Figure tend to disappear.

The numerical resolution chosen in this work is $(N, M, K) = (86, 16, 5)$ to match that used by Pringle & Kerswell in the $\theta$- and $z$-directions. This is sufficient to observe a drop-off of 6 decades in the axial spectrum and more than 8 decades in the radial and azimuthal energy spectra. The rapid drop-off in the axial wavenumbers is typical of lower-branch solutions and their weakly wiggling streak structure, it is even more pronounced as $Re \to \infty$ (Wang et al., 2007). The Navier-Stokes equations have been linearised around the TW solution expressed in the Galilean frame moving with speed $c$ along the axis. The corresponding eigenvalue problem was solved numerically using an Arnoldi routine for $Re$ up to 5000. There are 4 unstable eigenvalues (2 real and 2 complex conju-
Figure 4.3: Velocity profile of the asymmetric TW solution at the Hopf bifurcation ($\alpha = 0.75$, $Re = 1785$). Clockwise from top left: $z/L = 0, 0.25, 0.5, 0.75$. The arrows indicate the cross-stream velocity, and the shading indicates the difference between the streamwise velocity and the laminar profile (light/white indicating positive values and dark/red negative values).
gate ones) for \( Re < Re_H = 1785.6 \) and only 2 real ones for \( Re > Re_H \) indicating a Hopf bifurcation at \( Re = 1785.6 \) (see Figure 4.4). The center manifold theorem states that for \( Re \) close enough to \( Re_H \), a periodic orbit must exist in the same moving frame. Its amplitude scales locally like \( O(\sqrt{|Re_H - Re|}) \) (Kuznetsov, 2004). The period of the orbit is given by the imaginary part of the Hopf eigenvalue pair, here \( T = 2\pi/1m(\lambda_H) \sim 43 D/U \). A good estimation for the shifts is the distance travelled by the TW in the axial and azimuthal directions during this period, hence \( \Delta z^{(0)} = cT \sim 58.7 D \) and \( \Delta \theta^{(0)} = 0 \) (since this wave has no azimuthal phase speed). The starting point used for the Newton-Krylov algorithm is a slight perturbation of the TW along one of the two conjugate neutral directions, given by the Hopf eigenvector \( e_H \):

\[
(X^+)^{(0)} = \left( X_{TW} + \epsilon e_H, \Delta z^{(0)}, \Delta \theta^{(0)} \right).
\]  (4.4.19)

For values of \( \epsilon \sim O(10^{-2}) \) and \( Re_H - 1 < Re < Re_H \), the algorithm converged to a RPO, within a residual \( r_\infty = |X_{-\Delta z}(T) - X(0)|/|X(0)| \) of \( O(10^{-6}) \). The converged values of the period \( T \) and the shift \( \Delta z \) are very close to the expected values, while the RPO appears not to rotate (\( \Delta \theta = 0 \)). The new solution branch extends to \( Re < Re_H \) and we define the reduced positive parameter \( \delta := Re_H - Re \).

### 4.4.2 Continuation of the RPO

Continuation of these orbits along the \( Re \)-axis is used to produce a bifurcation diagram. Once one RPO is known at a given value of \( \delta \), it is straightforward to use it as an initial condition for the Newton-Krylov algorithm at slightly different values of \( \delta \). Progression towards larger \( \delta \) (i.e. lower \( Re \)) by using small enough steps enables one in principle to track the solution down until the branch possibly bends back.

Trajectories corresponding to RPO solutions have been traced on a two-dimensional plane \((D, E)\) for various values of \( \delta \). Here \( D := \int |\nabla \times \)
Figure 4.4: Stability analysis of the asymmetric TW for $\alpha = 0.75$ near the Hopf bifurcation point at $Re = Re_H = 1785.6$: growth rates (in units of U/D) against $Re$. Each line either indicates the locus of a real eigenvalue (solid) or a complex conjugate pair (dashed) as $Re$ changes. The blue (dark) lines correspond to those which are shift-&-reflect symmetric Wedin and Kerswell (2004) while the green (light) lines are anti-shift-&-reflect symmetric. The TW itself bifurcates off the mirror-symmetric TW branch near $Re = 1770$. 
Figure 4.5: Numerically found RPOs in the $(D,E)$ plane for different values of $\delta = 4$ (solid red) $\delta = 22$ (dotted blue) $\delta = 47$ (dashed green), with $\delta = Re_H - Re$. The asymmetric-symmetric (TWA) and mirror-symmetric (TWM) TW branches are also indicated (black dotted curves) parametrised by $Re$. The TWA at $Re = Re_H$ is in the centre of the $\delta = 4$ RPO loop. The RPO corresponding to $\delta = 47$ does not look perfectly closed, because of numerical issues detailed in Subsection 4.4.3. Note that the TWA branch exists only for $Re \geq 1770$ or $\delta \leq 15$.

$u'|^2d^3x$ and $E := \frac{1}{2} \int |u'|^2d^3x$ are respectively the dissipation and the kinetic energy of the disturbance $u' := u - u_{HP}$ to the laminar flow (see Figure 4.5). Such a projection makes these RPOs look periodic, whereas in the non-moving frame they are only relative periodic orbits. The orbits are slightly elliptic and correspond to a slow modulation of the TW from which they have bifurcated, whereas a TW at a given $Re$ would appear as a dot.

We define their normalized amplitude by:

$$\delta E = \frac{E_{\text{max}} - E_{\text{min}}}{E_{\text{max}} + E_{\text{min}}}$$

(4.4.20)

where the subscripts $\text{min}$ and $\text{max}$ denote extrema along a cycle of the RPO ($\delta E = 0$ for a TW solution). The further $\delta$ is away from the
bifurcation point, the larger the amplitude of the RPO, as is expected from a regular Hopf bifurcation. The values of $T$ and $\Delta z$ both keep the same order of magnitude. For $\delta$ up to 50, the tendency for the orbits is to achieve excursions towards regions of higher dissipation and energy. The energy variations along one cycle stay small for the range of $Re$ considered. Figure 4.6 shows the dependence of the orbit amplitude $\delta E$ on $Re$: $\delta E$ grows monotonically with the reduced parameter $\delta$ and is well approximated by the expression $\delta E = O(\delta^{\frac{1}{2}})$ near the bifurcation point.

The curve has been continued numerically until $\delta = 47$. Above this value, convergence becomes questionable - see Figure 4.6 - as discussed in Subsection 4.4.3 and in the conclusion.

The value of $\Delta z$ indicates the distance travelled by the exact coherent structure as it propagates down the pipe in time $T$. This defines a phase
speed for the RPO,

\[ c_{\text{RPO}} := \Delta z / T, \quad (4.4.21) \]

corresponding to the axial speed of the frame in which the orbit would be exactly periodic. Figure 4.7 is a diagram showing the phase velocity of both the asymmetric TW branch and the mirror-symmetric TW branch, compared to the phase velocity of the RPO which varies little over the range of \( Re \) considered. Note that the RPO exists at values of \( Re \) below that at which the asymmetric TW appears (see Figure 4.7).

Figure 4.8 shows the evolution of the velocity field in a cross-section moving with the speed \( c_{\text{RPO}} \) at \( \delta = 47 \) (i.e. \( Re \sim 1738 \)). This moving frame is chosen to emphasize slow-time variations. Near the left of the cross-section, the flow is strongly reminiscent of the pattern of the original asymmetric TW, which, of course, would give 4 identical slices. There is a slight periodic modulation of the shape and position of the
Figure 4.8: Velocity field of the RPO in the moving frame, at a cross-section defined by $z - c_{RPO}t = 0$ (see Eq. 4.4.21), for $\delta = 47$ ($Re = 1738$). From left to right, from top to bottom: $t = 7.2$, $t = 18.7$, $t = 30.3$ and $t = 41.8$ in units of $D/U$. The colours and arrows are as in Figure 4.3.
low-speed streak. The high-speed streaks close to the wall look more robust in shape during a cycle, but their intensity fluctuates. As $\delta$ increases, the intensity inside the patches of axial velocity on the right of Figure 4.8 fluctuates more in time than the ones of the left. The larger $\delta$, the more symmetric the cross-sectional pattern looks (this feature is also shared with the asymmetric TW branch as it approaches the mirror-symmetric branch). The cross-section velocity is more steady than the axial velocity, except in the vicinity of the weaker low-speed streak on the right of the pipe, where the vortex looks ‘attached’ to the streak.

Figure 4.9 displays the friction factor $\Lambda$, defined using dimensional variables as

$$\Lambda := \frac{dp}{dz} \left(\frac{1}{2} \rho U^2\right),$$

(4.4.22)
of the RPO branch. For any given value of $Re$, $\Lambda$ along a cycle is always below those of the asymmetric and mirror-symmetric TW solutions. The highest value $\Lambda_{\text{max}}$ reached during a cycle gets closer to that of the mirror-symmetric branch as $Re$ is decreased.

### 4.4.3 Instability of the RPO and Numerical Issues

While performing the continuation in $\delta$, we observed that $r_\infty$ (the value around which the numerical residual stagnates) increases steadily with $\delta$. This trend is clear from Figure 4.6. While animation of the velocity field does not show any discontinuity in time for $\delta > 25$, the projection on the $(D, E)$ plane no longer looks perfectly closed (see Figure 4.5). We deliberately stopped the continuation near $\delta \sim 50$ when it was difficult to reduce $r_\infty$ below $10^{-2}$.

In order to understand this behaviour of $r_\infty$, we first need an idea of how accurate the search for RPOs is expected to be. The minimum residual $r_\infty$ is limited by both the accuracy of the time-stepping scheme and the natural instability of the trajectory. Any component of any point on the numerically found RPO is accurate to at least $O(\epsilon_m)$, the machine
The precision. After a period $T$, this numerical discrepancy has increased by a factor $e^{\lambda T}$, with $\lambda$ being now the largest Floquet exponent of the RPO. Indeed a useful rule-of-thumb for estimating the lowest residual attainable is:

$$r_\infty \sim O\left(ne_\text{m}e^{\lambda T}\right). \tag{4.4.23}$$

We can illustrate this formula by simply time-stepping from a sufficiently converged asymmetric TW at $\delta = 0$. After the TW has travelled one pipe length, $r(t = L/c) \sim 10^{-10}$; after a time $T$ (the period of the RPO), $r(T) \sim 10^{-5}$. Using expression (4.4.23), we can estimate $\lambda_{TW} \sim \ln(10^5)/T \sim 0.26 U/D$. Sufficiently close to the Hopf bifurcation, both the RPO and the TW are expected to have the same leading Floquet exponent $\lambda$, hence for the RPO we get the estimation $\lambda_{RPO}(\delta = 0) \sim 0.26 U/D$. Using the data in Figure 4.6, since the period $T$ does not vary much (less than 1%) within the range of $\delta$ considered, we can deduce
a scaling for the Floquet exponent of the RPO: \( \lambda_{RPO}(\delta) \sim O(\delta^{4}) \). Hence the RPO is more unstable away from the bifurcation point. A possible explanation is that the streaks (whose strong transversal velocity gradients are already responsible for the inertial instability of the TW), are here subject to stronger slow-time modulation as \( \delta \) increases. This harmonic modulation of the streaky field over a full cycle is likely to increase the instability of the whole flow.

### 4.5 Connection with a TW Solution

A leading question is how the RPO found in this paper is related to transition to turbulence in a pipe. The asymmetric TW is already known to play a special role: firstly, it lies on the laminar-turbulent boundary (also called the ‘edge of chaos’ (Eckhardt et al., 2007; Schneider et al., 2007b)), which is the set of initial conditions separating trajectories which directly relaminarise from those which lead to turbulent behaviour. This means that infinitesimal perturbations to the TW can either lead to transition or relaminarisation, depending on their exact form. Secondly, recent studies have shown that for analogous parameters, but larger \( Re \), this asymmetric TW is generally the only exact travelling wave solution closely visited by trajectories when constrained on the laminar-turbulent boundary (Duguet et al., 2008b). It is thus interesting to see if other kind of exact recurrent states, like the RPO found here fulfil the expectation of: (a) lying on the laminar-turbulent boundary as well; (b) connecting to other states via heteroclinic connections; and (c) getting transiently approached by transitional trajectories.

We have chosen to concentrate on \( \delta = 2 \ (Re = 1783.5) \), and perturbed the RPO by two random vectors in the following manner:

\[
X(t = 0) = X_{RPO} + \epsilon (\cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_2).
\]  

(4.5.24)
In this expression, $\epsilon$ is a small parameter (here $10^{-4}$), $e_1$ and $e_2$ are two random vectors of norm unity, which are generic enough to possess non-zero components along the unstable manifold of the RPO. $\phi$ is a shooting angle which has been varied in the range $[0, 2\pi]$. We notice that several trajectories starting from the new state $X(t = 0)$ relaminarise directly while other trajectories undergo a short ‘turbulent’ transient, indicated by a dramatic rise in the kinetic energy and dissipation of the disturbance to the laminar flow, and by a loss of symmetry of the flow. Hence the RPO found for $Re = 1783.6$ is also on the laminar-turbulent boundary.

Establishing heteroclinic connections between the RPO and other exact coherent structures is an involved and technically demanding task. We can, however, seek suggestive evidence for such links by exploring the evolution of trajectories originating close to the RPO and confined to remain in the laminar-turbulent boundary. This was attempted for the RPO at $\delta = 2$ by refining the angle $\phi$ defined in (4.5.24) to find a phase-space trajectory which stays near the laminar-turbulent boundary, i.e. neither directly relaminarises nor undergoes a turbulent transient. This is done via shooting method based on bisection of the value of $\phi$. Refining $\phi$ up to three significant digits results in a trajectory called $H_1$, which eventually relaminarises after a significantly long transient along the laminar-turbulent boundary. For a short duration during the transient (less than $10 D/U$), all velocity components oscillate with the same apparent period on a short-time scale, and the energy transiently reaches a plateau. Based upon previous experience (Duguet et al., 2008b), we recognised this as the signature of a travelling wave solution lying nearby in phase space. The scalar function

$$r_{\min}(t) = \min_{t' > t} \{ r_t(t'), |\partial r_t / \partial t'| = 0 \} \quad (4.5.25)$$
where

$$r_t(t' > t) = \frac{|X(t') - X(t)|}{|X(t)|}. \quad (4.5.26)$$

was calculated along the whole trajectory. This function, closely linked to $|G|$, measures how recurrent a flow is at a given location in space (ignoring the possibility of shifted recurrences). Figure 4.10 shows $r_{\text{min}}$ as a function of the starting point on the edge trajectory $H_1$. While $t \leq 60 \, D/U$, $r_{\text{min}}$ is very low ($\mathcal{O}(10^{-3})$) and the flow is nearly recurrent. The slow-time modulation reflects the modulation of the flow along the RPO. Later the trajectory leaves the neighbourhood of the RPO because of its instability, and $r_{\text{min}}$ increases up to large values of $\approx 0.2$. At a later phase corresponding to the plateau in energy, $r_{\text{min}}(t)$ displays a clear dip (labelled N1 in Figure 4.10) down to values of $\approx 5 \times 10^{-2}$, and then increases again. Such low values of $r_{\text{min}}$ are never reached if the angle $\phi$ is chosen randomly and indicate that the edge trajectory has entered the (phase-space) vicinity of an exact periodic solution, at which $r_{\text{min}}$ vanishes. The starting point at $t \sim 150$ which yielded the lowest $r_{\text{min}}$ was used as an initial state for the Newton-Krylov algorithm with $\Delta z = 2\pi/\alpha$ and $\Delta \theta = 0$. This readily converged to the asymmetric TW, shifted azimuthally by a half turn, with $r_{\text{min}} \sim 10^{-10}$.

Shooting in the opposite direction, then refining properly the angle $\phi$, leads to another ‘edge’ trajectory $H_2$. The corresponding recurrence function $r_{\text{min}}(t)$ is plotted in Figure 4.10. The features of $r_{\text{min}}$ are reminiscent of those of the trajectory $H_1$. A dip (labelled N2) appears near $t \approx 190 \, D/U$, and the corresponding state has been used as a starting point for the Newton-Krylov algorithm (again we look explicitly for a TW solution). The algorithm has converged to another TW solution, that we call $2b_{1.5}$. Close examination revealed that it is equivalent of the TW $2b_{1.25}$ mentioned in Kerswell and Tutty (2007) and Duguet et al. (2008b), but with an axial wavenumber of $\alpha = 1.5$ instead of $94$. 

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Figure 4.10: Recurrence function $r_{\text{min}}$ vs. time for the two ‘edge’ trajectories $H_1$ (red solid) and $H_2$ (blue dotted), starting from the perturbed RPO and constrained to the laminar-turbulent boundary. The dip labelled N1 (resp. N2) near $t \sim 150$ (resp. $t \sim 190$) indicates an approach towards a TW state.

$\alpha = 1.25$. Here $\alpha = 0.75 = 1.5/2$, so that exactly two wavelengths of the TW fill the whole domain. This TW solution was already mentioned in Duguet et al. (2008b) because it is an attractor for edge trajectories when $Re = 2400$, $\alpha = 1.25$, and when the flow is constrained to a 2-fold rotationally symmetric subspace. This attracting property is lost when no rotational symmetry is forced, which explains why the TW here first attracts and then repels the edge trajectory $H_2$.

Given the difficulty of getting the Newton-Krylov algorithm to converge in high dimensions unless the starting guess is sufficiently close to a solution, it is reasonable to assume that the above trajectories enter the neighbourhood of the TWs before ultimately relaminarising. Since the unstable manifold of these TWs in the laminar-turbulent boundary is of such small dimension, it is tempting to speculate that these visits are actually indicative of heteroclinic connections linking the RPO to the two TWs (see Figure 4.11). Establishing this formally is a signifi-
cant undertaking not pursued further here but the idea that trajectories link different coherent structures to produce a saddle structure on the laminar-turbulent boundary is consistent with other recent observations (Duguet et al., 2008b).

The trajectories $H_1$ and $H_2$ are also displayed in $(D, E)$ coordinates in Figure 4.12. In both cases the trajectory escapes from the RPO solution along its unstable manifold, in the direction locally tangent to the laminar-turbulent boundary. Both reach very large values of dissipation and energy, before turning back to the laminar state and going through the vicinity (in terms of $r_{\text{min}}$) of one of the TWs. The use of the quadratic quantities $E$ and $D$, however, does not make this approach so apparent and the time of closest approach in the $(D, E)$ plane does not coincide with that according to $r_{\text{min}}$ (e.g. $N_2$ on H2 is not the closest point of approach to $29.1.5$ in the $(D, E)$ plane). This difference, of course, highlights the ongoing dilemma of how to select the ‘right’ norm to measure distances in phase space for the spatial-temporal pdes which are the Navier-Stokes equations.

### 4.6 Conclusion

We have shown that it is possible to numerically capture relative periodic orbits (RPOs) in pipe flow when the associated dynamical system size exceeds $10^5$ degrees of freedom. We have applied successfully our method to find a RPO branch in the vicinity of the Hopf bifurcation occurring along the asymmetric Travelling Wave branch for $\alpha = 0.75$. This exact coherent structure consists of a streaky pattern which is modulated over one period of roughly $43 \, D/U$, while travelling approximately 60 pipe diameters downwards in the axial direction.

Further evidence is presented here that the laminar-turbulent boundary is structured around of network of exact solutions linked to each other by heteroclinic and homoclinic connections (Kerswell and Tutton,
Figure 4.11: Phase-space sketch of the network of the possible dynamical connections between the asymmetric TW (\textit{TWA}), the TW $2b_{1.5}$ and the RPO, detected for $\delta = 2$. All these trajectories lie on the laminar-turbulent boundary.

Figure 4.12: $(D - E)$ projection of the ‘edge’ trajectories $H_1$ (red) and $H_2$ (blue) starting from the perturbed RPO. Also shown are the TW states identified using the Newton-Krylov algorithm: TWA stands for the Asymmetric TW, TW2b for the TW solution $2b_{1.5}$. 
2007; Wang et al., 2007; Gibson et al., 2008; Duguet et al., 2008b). For \( L \sim 5D (\alpha = 0.625) \) and \( Re = 2875 \), asymmetric TW solutions are recurrently approached by laminar-turbulent trajectories (Duguet et al., 2008b): here for \( L \sim 4.18D (\alpha = 0.75) \) and \( Re = 1783.6 \), the same conclusion seems to hold. However, now this schematic view of edge trajectories recurrently visiting neighbourhoods of TWs needs to be extended to take into account relative periodic orbits. All these solutions and their stable manifolds make up the laminar-turbulent boundary.

The main problem in accurately identifying RPOs is the exponential divergence of a nearby orbit from the RPO. In practice, the instability of the RPO together with its typically long period lead to a large value for its leading characteristic multiplier. This causes a significant loss of numerical accuracy when searching for neighbouring orbits which almost close on themselves. This effect was found to worsen away from the bifurcation point as the RPO becomes more unstable so that it was only possible to trace the RPO branch accurately in a relatively small neighbourhood of the bifurcation point. As a result, its amplitude remains small and the RPO dynamics are close to that of the underlying TW solution. The original goal of tracing the RPO down (in \( Re \)) to a likely saddle node bifurcation point and then exploring the ‘other’ branch have, unfortunately, been out of reach. Whether this ‘other’ branch reconnects to another TW branch or becomes a more nonlinear unstable object possibly embedded in the flow’s turbulent dynamics remains a challenge for the future. There are ways to improve the numerical algorithm such as a multiple-shooting technique where the orbit is subdivided into manageable pieces which now clearly need to be explored.

Physically, the instability of the RPO means that phase-space trajectories are unlikely to spend a long time in its vicinity, despite the likelihood that the RPO is linked via heteroclinic connections to other more attracting solutions. In particular, the probability is low that such a solution could be identified experimentally, despite recent progress made
in flow visualisation or indeed numerically as part of a direct numerical simulation. The same conclusion may well apply to all RPOs of the system.

At the start of this study, the stability of various lower TW branches was considered in order to identify how frequently Hopf bifurcations of TW branches arise. For $\alpha = 0.75$, the asymmetric TW branch undergoes only one Hopf bifurcation below $Re = 5000$ which is examined in this paper. The mirror-symmetric branch displays another one near $Re = 3487$, its period near the bifurcating point is $T = 68$, which makes it even more unstable and harder to continue than the previous one. For the same parameters, we have checked that another known branch of TW (referred to as $2b_{1.25}$ (Kerswell and Tutty, 2007; Duguet et al., 2008b)) does not display any Hopf bifurcations at all below $Re = 3400$. Despite the fact that many more TWs might exist, it appears that Hopf bifurcations of lower-branch TW solutions are not frequent, and that short-period RPOs associated with them are not generic objects. From this point of view, the fact that mainly TW solutions have been identified during transition does not mean that RPOs do not participate in the transitional dynamics, but rather that their contribution is likely to be more infrequent and fleeting. However, it might not be the case for other shear flows such as plane Couette flow where (non-relative) periodic orbits also bifurcate from a lower-branch steady state (Kawahara and Kida, 2001; Kawahara, 2005; Wang et al., 2007). This does not exclude either the existence of RPO solutions entirely disconnected from the TW solutions, which would appear via saddle-node bifurcations. Such a (periodic) solution has been computed in plane Couette flow, and its dynamics is embedded in the turbulent dynamics.Kawahara and Kida (2001); Viswanath (2007) The numerical method described in Section 4.3 is an adequate tool to seek them providing a good enough starting point is identified for the Newton-Krylov algorithm.
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Chapter 5

Simulating Pipe Flow

5.1 Time-Stepping the Navier-Stokes Equations

5.1.1 The Equations of Incompressible Fluid Flow

Numerically simulating the flow of fluids has naturally been of great interest for as long as computers have been available. Unsurprisingly, this is a task with numerous subtleties which require careful consideration. The first decision that needs to be made is how to formulate the underlying equations. The naïve approach is termed primitive variables - one simply uses the natural quantities $u$ and $p$ as described by the equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u$$

$$\nabla \cdot u = 0.$$  

Conceptually straightforward, this method quickly runs into difficulties. The fundamental idea of any simulation code is that given $(u, p)(x, t)$, one can then calculate $(u, p)(x, t + dt)$. The evolution of $u$ is given by equation (5.1.1), however we have no such evolution equation for the pressure. This can be alleviated by taking the divergence of equation

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(5.1.1) and substituting in (5.1.2) to get

\[
\frac{1}{\rho} \nabla^2 p = -\nabla \cdot (u \nabla u) \tag{5.1.3}
\]

as an equation for the pressure field. This is all well and good until we observe that there are no boundary conditions for \( p \), and so we must either fabricate them (a non-trivial task), or look for an alternative formulation.

As it is pressure that is causing all our problems, the natural step to take is to eliminate it from the equations. This is simply achievable by taking the curl of the Navier-Stokes equations, leading to the vorticity-transport equations

\[
\frac{\partial \omega}{\partial t} + (u \nabla) \omega - (\omega \nabla) u = \nu \nabla^2 \omega \tag{5.1.4}
\]

where

\[
\omega = \nabla \times u \tag{5.1.5}
\]

is the vorticity. Equation (5.1.4) must then be supplemented with the additional equation

\[
\nabla^2 u = -\nabla \times \omega \tag{5.1.6}
\]

found by taking the curl of the definition of the vorticity. Although equations (5.1.4) and (5.1.6) are a complete formulation giving the evolution of the flow, we note we are still short of boundary conditions - those for the vorticity field. Again how one can find these is not obvious.

The idea of eliminating pressure from the equations governing fluid flow is a persistent one and appears in the majority of techniques, such as stream-function representations, poloidal-toroidal decompositions, et cetera. The problem of boundary conditions remains.
5.1.2 The Fractional Step Method

For our purposes we wish to use a formulation that is as generalisable and retains as much physical meaning as possible. As such we remain firmly with primitive variables and adopt an algorithm known as the Fractional Step Method. This procedure dates back to Chorin (1968) and Temam (1969) and, as the name implies, breaks the evolution of the variables at each time interval down into two substeps. At its most general, and assuming no-slip boundary conditions, the first step is to solve

\[
\frac{\tilde{u}^{n+1} - u^n}{\Delta t} = - (u^n \nabla) u^n + \nu \nabla^2 u^n \tag{5.1.7}
\]

\[
\tilde{u}^{n+1}|_{\partial \Omega} = 0, \tag{5.1.8}
\]

where \( \Omega \) is the domain with boundary \( \partial \Omega \) being simulated and the superscript \( n \) is the time interval, \( t_n \). The new velocity field \( \tilde{u}^{n+1} \) has been generated without reference to a pressure field so will in general not be incompressible. The second half of the fractional step is to adjust this by projecting the velocity field onto an incompressible subspace. This is done via

\[
\frac{u^{n+1} - \tilde{u}^{n+1}}{\Delta t} = \nabla \Pi^{n+1} \tag{5.1.9}
\]

\[
\nabla . u^{n+1} = 0, \quad \frac{\partial u^{n+1}}{\partial n} \bigg|_{\partial \Omega} = 0. \tag{5.1.10}
\]

The new velocity field \( u^{n+1} \) is by construction incompressible but we have only imposed a normal velocity boundary condition rather than the full non-slip condition. This is because \( \Pi^{n+1} \) is calculated from the Poisson equation

\[
\nabla^2 \Pi^{n+1} = \frac{\nabla \cdot \tilde{u}^{n+1}}{\Delta t}, \quad \frac{\partial \Pi^{n+1}}{\partial n} \bigg|_{\partial \Omega} = 0. \tag{5.1.11}
\]
and as such can only support one boundary condition. We refer to \( \Pi \) as
the pseudo-pressure, and it is crucially different from the actual pressure.

The essence of the method we have laid out is that at each time-interval we calculate two different velocity fields. One, \( \tilde{u} \), is a no-slip but compressible flow, the other, \( u \), is an incompressible flow which has
tangential flow at the boundary. Although the majority of papers using
this technique present \( u \) as their velocity field neither is more accurate
than the other.

### 5.1.3 Non-Dimensionalisation

Non-dimensionalising the equations to be solved is not strictly necessary
before simulating them, however it is in general advisable. Firstly, at
least in the case of pipe flow, it reduces the number of parameters to
be varied down to just the Reynolds number. Secondly, as all quantities
should be roughly unitary, it makes debugging easier to spot - a velocity
component \( \sim 10^{23} \) is likely the result of the dividing by zero. It is
important to choose your non-dimensionalising carefully. The travelling
wave code considered flow driven by a constant pressure gradient along
the pipe and so was non-dimensionalised by

\[
\begin{align*}
    u^* &= U_{cl} u_p \quad (5.1.12) \\
    s^* &= s_0 s_p \quad (5.1.13) \\
    Re_p &= \frac{U_{cl} s_0}{\nu} \quad (5.1.14)
\end{align*}
\]

where \( U_{cl} \) is the centreline velocity of the laminar flow, \( s_0 \) is the pipe
radius and \( x^* \) is the dimensional version of \( x \). We, however, wished to
simulate flow in a pipe with a constant mass-flux, and so instead chose
the non-dimensionalisation

\[
\begin{align*}
    u^* &= \bar{W} u_m \quad (5.1.15) \\
    s^* &= s_0 s_m \quad (5.1.16)
\end{align*}
\]
\[ Re_m = \frac{2W_s}{\nu} \]  
\[ \text{(5.1.17)} \]

with \( W \) being the mean axial velocity. The factor of 2 is included in the definition of the Reynolds number so that for laminar flow \( Re_p = Re_m \).

We can convert between the two non-dimensionalisations using

\[ u_m = 2 \frac{Re_p}{Re_m} u_p \]  
\[ p_m = 4 \frac{Re_p^2}{Re_m^2} u_m \]  
\[ t_m = \frac{1}{2} \frac{Re_m}{Re_p} \]  
\[ \text{(5.1.18)} \]
\[ \text{(5.1.19)} \]
\[ \text{(5.1.20)} \]

From here on we shall drop the subscript \( m \). Applying this to the momentum equation (5.1.1) gives

\[ \frac{\partial u^*}{\partial t} + (u^* \cdot \nabla) u^* = -\frac{1}{\rho} \nabla p^* + \nu \nabla^2 u^* \]  
\[ \Rightarrow \frac{W^2}{s_0} \left[ \frac{\partial}{\partial t} + (u \cdot \nabla) \right] u = -\frac{1}{\rho} \frac{W}{s_0} \nabla p + \nu \frac{W}{s_0} \nabla^2 u \]  
\[ \Rightarrow \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + 2 Re \nabla^2 u \]  
\[ \text{(5.1.21)} \]
\[ \text{(5.1.22)} \]
\[ \text{(5.1.23)} \]

with the incompressibility condition unchanged as

\[ \nabla \cdot u = 0. \]  
\[ \text{(5.1.24)} \]

### 5.1.4 A Second Order Time Stepping Routine

Lopez et al. (2002) suggest the method

- \[ \frac{1}{\Delta t^3} [3\tilde{u}^{n+1} - 4u^n + u^{n-1}] = -\nabla p^n - 2N^n - N^{n-1} + \frac{2}{Re} \nabla^2 u^{n+1} \]  
  subject to \( \tilde{u}^{n+1}|_{\partial \Omega} = 0 \)

- \[ -\nabla^2 \Pi^{n+1} = -\frac{3}{2 \lambda} \nabla \cdot \tilde{u}^{n+1} \]  
  subject to \( \frac{\partial}{\partial n} \Pi^{n+1}|_{\partial \Omega} = 0 \)

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\[ p^{n+1} = \Pi^{n+1} + p^n + \frac{2}{Re} \nabla \cdot \tilde{u}^{n+1} \]
\[ u^{n+1} = \tilde{u}^{n+1} - \frac{2 \Delta t}{3} \nabla \Pi^{n+1} \]

where \( N^n = [u(t_n) \cdot \nabla] u(t_n) \). The inclusion of the divergence of the velocity field in updating the pressure field in the third step does not improve the formal accuracy of the code. Instead it simply means that \( p \) satisfies a consistent Neumann boundary condition. The method is second-order in time and relatively simple to implement.

As noted by Guermond et al. (2006), \( \tilde{u} \) and \( u \) are linked in a trivial manner in the final step. This can be used to eliminate \( \tilde{u} \) from the method, in which case only one velocity field need be stored - the no-slip compressible field. The method we used was essentially that of Lopez et al., but with this simplification applied. We also dropped the ‘compressibility correction’ to the pressure field, and treated the viscous term with Crank-Nicolson. This gives the final three step method:

\[ \left( \frac{3}{2\Delta t} - \frac{1}{Re} \nabla^2 \right) u^{n+1} = \frac{1}{2\Delta t} [4u^n - u^{n-1}] - \frac{4}{3} \nabla \Pi^n + \nabla \Pi^{n-1} \]
\[ + \frac{1}{Re} \nabla^2 u^n - \nabla p^n - [2N^n - N^{n-1}] \]
subject to \( \tilde{u}^{n+1}|_{\partial \Omega} = 0 \)

\[ -\nabla^2 \Pi^{n+1} = -\frac{3}{2\Delta t} \nabla \cdot u^{n+1} \]
subject to \( \frac{\partial}{\partial n} \Pi^{n+1}|_{\partial \Omega} = 0 \)

\[ p^{n+1} = \Pi^{n+1} + p^n. \]

### 5.1.5 Spectral Representation

In order to maximise the compatibility of the code with the existing Travelling Wave code, we choose an almost identical formulation
where the coefficients \( x_{mn} \) are functions of time and the radial functions are defined by

\[
\Theta_n(s;i) := \begin{cases} T_{2n+2}(s) - T_{2n}(s) & \text{i odd,} \\ T_{2n+3}(s) - T_{2n+1}(s) & \text{i even,} \end{cases} \tag{5.1.26}
\]

\[
\Phi_n(s;i) := \begin{cases} T_{2n+3}(s) - T_{2n+1}(s) & \text{i odd,} \\ T_{2n+2}(s) - T_{2n}(s) & \text{i even,} \end{cases} \tag{5.1.27}
\]

\[
\Psi_n(s;i) := \begin{cases} s^2T_{2n+1}(s) & \text{i odd,} \\ T_{2n}(s) & \text{i even.} \end{cases} \tag{5.1.28}
\]

Again, \( T_n \) is the \( n \)th Chebychev function, \( \alpha \) is the axial wavenumber and \( m_0 \) is the azimuthal wavenumber. The only important difference is that now a factor of \( s^2 \) is included in the pressure functions for odd \( m_0 \).

### 5.1.6 Mass Flux

As already noted we wish to simulate flow in a pipe subject to a constant mass flux along a pipe. The complexity that this simple statement adds is that given a fixed pressure gradient along the pipe, the onset of turbulence leads to a decrease in mass flux. We therefore have to compensate for this by increasing the pressure gradient in such a way as to precisely offset this flow rate drop. We introduce the notation

\[
(\cdot) = \int_0^{2\pi/\alpha} \int_0^{2\pi} \int_0^1 (\cdot) sdsd\phi dz \tag{5.1.29}
\]
and calculate the effect of this on the $z$-component of the Navier-Stokes equations (we are considering not the full flow field, but instead the perturbation to the laminar state $u_{larm} = 2(1 - s^2)\hat{z}$)

$$\frac{\partial}{\partial t} \langle w \rangle + \langle (u, \nabla)w \rangle = \langle -\frac{\partial}{\partial z} p \rangle + \langle \frac{Re}{2} \nabla^2 w \rangle. \quad (5.1.30)$$

We shall be using a spectral code with Fourier modes in the two periodic directions. We thus consider the flow as the superposition

$$u = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} u_{ml}(s, t)e^{im0\phi}e^{ialz}, \quad (5.1.31)$$

$$p = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} p_{ml}(s)e^{im0\phi}e^{ialz}. \quad (5.1.32)$$

This is actually not quite correct because we do not actually require that the pressure field is periodic - simply that the gradient of it is. Thus we can write $p = \lambda z + \hat{p}$ where $\hat{p}$ is the periodic part and $\lambda$ is a constant. Thus

$$\langle \frac{\partial}{\partial z} p \rangle = \langle \lambda + \frac{\partial}{\partial z} \hat{p} \rangle \quad (5.1.33)$$

$$= \int \int \int \int \int \lambda sd\phi dz + \int \int \frac{\partial}{\partial z} \hat{p} sd\phi dz \quad (5.1.34)$$

$$= \frac{4\pi^2}{2} \lambda \frac{s^2}{\alpha} \int_{s=0}^{s=1} \int \int \int \hat{p} sd\phi dz \quad (5.1.35)$$

$$= \frac{2\pi^2}{\alpha} \lambda \quad (5.1.36)$$

$$\langle (u, \nabla)w \rangle = \int \int \int sdsd\phi dz \left[ \frac{\partial w}{\partial s} + \frac{\partial w}{\partial \phi} + \frac{\partial w}{\partial z} \right] \quad (5.1.37)$$

$$= \int \int sdsd\phi dz \left[ \nabla \cdot (uw) - w \nabla \cdot u \right] \quad (5.1.38)$$

$$= \oint \hat{u} wds \quad (5.1.39)$$

$$= 0 \quad (5.1.40)$$
\[
\frac{2}{Re} \left\langle \nabla^2 w \right\rangle = \frac{2}{Re} \int \int \int \nabla (\nabla w) s ds d\phi dz \quad (5.1.41)
\]
\[
= \frac{2}{Re} \int \nabla w dS \quad (5.1.42)
\]

The velocity field \( w \) is the same at either end of the pipe, but the surface normal is opposed, so they cancel out. On the surface of the pipe \( \partial w / \partial \phi = \partial w / \partial z = 0 \). Further any periodic terms integrate to zero. We are thus left with
\[
\left( \frac{2}{Re} \nabla^2 w \right) = \frac{2}{Re} \int \int \left. \frac{\partial w}{\partial s} \right|_{s=1} d\phi dz \quad (5.1.43)
\]
\[
= \frac{8\pi^2}{\alpha Re} \left. \frac{\partial w_0}{\partial s} \right|_{s=1}. \quad (5.1.44)
\]

For there to be no change in mass flux we need \( \frac{\partial}{\partial t} \langle w \rangle = 0 \) which requires the balance
\[
\frac{2\pi^2}{\alpha} \lambda = \frac{8\pi^2}{\alpha Re} \left. \frac{\partial w_0}{\partial s} \right|_{s=1} \quad (5.1.45)
\]
\[
\Rightarrow \lambda = \frac{4}{Re} \left. \frac{\partial w_0}{\partial s} \right|_{s=1}. \quad (5.1.46)
\]

### 5.2 Testing the Code (Part 1)

#### 5.2.1 Linear Pipe Flow

The simplest published results that we can test our code against are those for the linear problem. In this case it is well known that the flow is stable and any perturbation will decay. The different Fourier-modes do not interact and the energy in each mode decays away independently of the others. In the large time limit the decay is exponential and governed for each mode by the least decaying eigenvalue. In the case of \( Re = 3000 \) and \( \alpha = 1 \), this was calculated by Schmid and Henningson (1994). With a fixed radial resolution we calculated the decay rate of the \( m = 0, 1, 2, 3 \) and 4 modes with \( \alpha = 1 \) for two different time step sizes - \( dt = 5 \times 10^{-4} \)
Table 5.1: We compare the rate of decay of a random perturbation in a linear setting for varying azimuthal wave number ($m_0$). The Reynolds number was fixed at 3000, and in each case one streamwise mode, the $\alpha = 1$ mode, was considered. The published results of Schmid and Henningson (1994) are compared to those produced by our code with 25 radial modes for two different sizes of time step in units of $D/U$.

and $5 \times 10^{-5}$ (units of $D/U$). The results are shown in table 5.1, revealing good agreement.

### 5.2.2 Nonlinear Pipe Flow

Having confirmed that the linear parts of the simulation work correctly, the next step is to test the nonlinear terms. This was done by comparing our code to that created independently by A.P. Willis. The methodology of that code is described in Willis and Kerswell (2009). Both codes took the known travelling wave S3, with $\alpha = 2.44$ and $Re = 1800$ as a starting condition. It was perturbed by immediately reducing the Reynolds number to 1750. This results in the perturbation decaying away, and then the total energy and wall shear stress of the perturbation was compared between the codes. The results plotted in figure 5.1 show strong agreement between the codes.

### 5.3 Transient Growth

#### 5.3.1 On the Monotonicity of Decay

When dealing with systems of differential equations, typically if all the associated eigenvalues are negative then any initial condition will decay monotonically. However, this is because most of the time an unmen-
Figure 5.1: The decay of the travelling wave S3. The solution was placed into both the code presented here and that created by A.P. Willis. It was perturbed by reducing the Reynolds number from 1800 to 1750 and the evolution was plotted. The two codes agree well both in terms of total energy and wall shear stress.
tioned caveat is included - that the operator is ‘normal’. That is to say, all of its associated eigenfunctions are orthogonal to one another.

Let us suppose that we have a linear differential operator $\mathcal{L}$ and that under the inner-product $\langle \quad , \quad \rangle$ it has orthonormal eigenfunctions $\psi_1, \psi_2, \psi_3, \ldots$, each with associated eigenvalue $\lambda_1, \lambda_2, \lambda_3, \ldots$. Then any initial condition can be written as a weighted sum over the eigenfunctions

$$u = \sum_{i=1}^{\infty} a_i \psi_i \quad (5.3.47)$$

and further its evolution is simply given as

$$u = \sum_{i=1}^{\infty} e^{\lambda_i t} a_i \psi_i. \quad (5.3.48)$$

We can then denote the ‘energy’ of the state as

$$\langle u^2 \rangle = \langle \sum_{i=1}^{\infty} e^{\lambda_i t} a_i \psi_i, \sum_{i=1}^{\infty} e^{\lambda_i t} a_i \psi_i \rangle \quad (5.3.49)$$

$$= \sum_{i=1}^{\infty} e^{2\lambda_i t} a_i^2 \quad (5.3.50)$$

due to the orthonormality of the eigenfunctions. We plainly see that if all the eigenvalues are negative then the energy monotonically, and indeed exponentially, decays. It is also apparent that if the eigenfunctions are not all orthogonal then something more complicated can occur.

This is nicely illustrated by the toy model presented by Trefethen et al. (1993). Consider the system of equations

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1/Re & 0 \\ 1 & -2/Re \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \quad (5.3.51)$$

where $Re$ is a large parameter equivalent to the Reynolds number. The
setup can be easily solved giving eigenvalues and normalised eigenvectors

\[
\begin{align*}
\lambda_1 &= -1/Re, \quad \phi_1 = \frac{1}{\sqrt{1 + Re^2}} \begin{bmatrix} 1 \\ Re \end{bmatrix} \\
\lambda_2 &= -2/Re, \quad \phi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{align*}
\] (5.3.52) (5.3.53)

The eigenvalues are both negative, so we know that for large time all disturbances will decay asymptotically to zero. However, the two eigenvectors are non-orthogonal (unsurprising as the RHS matrix was also non-normal) allowing different short term behaviour. Taking an example initial condition of \((u, v) = (1, 0)\) we get

\[
\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 1 \\ Re \end{bmatrix} e^{-t/Re} - \begin{bmatrix} 0 \\ Re \end{bmatrix} e^{-2t/Re}
\]

\[= \begin{bmatrix} 1 - t/Re + \mathcal{O}(t/Re)^2 \\ t + \mathcal{O}(t/Re)^2 \end{bmatrix}.\] (5.3.54) (5.3.55)

This solution will exhibit algebraic growth until \(t = \mathcal{O}(Re)\), and further it does not just amplify the initial state but instead it twists as the \(u\)-component becomes smaller and the \(v\)-component grows.

Returning to fluid dynamics, we now consider the problem of pipe flow linearised around the laminar state. Clearly this represents a non-normal operator and so it allows the possibility of perturbation growth, even while remaining linearly stable. The question to ask is what initial condition leads to greatest amount of growth. As the growth in energy must be temporary, it is called \textit{transient} growth.

### 5.3.2 Formulating Transient Growth

Let us begin by considering our goals. We wish to calculate the precise form of initial condition that leads to the greatest increase in perturbation energy before it subsequently decays. This is simply represented by
attempting to maximise
\[
\mathcal{L} = \frac{\langle \mathbf{u}(\mathbf{x}, T) \rangle^2}{\langle \mathbf{u}(\mathbf{x}, 0) \rangle^2}
\] (5.3.56)

(recall we have defined the inner product as the volume integral over the periodic domain being considered). However, we want to maximise this subject to three constraints: that the flow at all times satisfies and evolves as described by the Navier-Stokes equation; that the flow is incompressible at all times; and that there is no change in mass flux along the pipe at any moment. These constraints can be imposed using Lagrange multipliers, giving the functional

\[
\mathcal{L} := \frac{\langle \mathbf{u}(\mathbf{x}, T) \rangle^2}{\langle \mathbf{u}(\mathbf{x}, 0) \rangle^2} - \int_0^T \langle \nu, \left[ \frac{\partial \mathbf{u}}{\partial t} + 2(1 - s^2) \frac{\partial \mathbf{u}}{\partial z} - 4su\hat{z} + \nabla p - \frac{2}{Re} \nabla^2 \mathbf{u} \right] \rangle dt
- \int_0^T \langle \Pi \nabla \cdot \mathbf{u} \rangle dt
- \int_0^T \Gamma(t) \langle \mathbf{u} \cdot \hat{z} \rangle dt.
\] (5.3.57)

The quantities \(\nu, \Pi\) and \(\Gamma\) are known as the adjoint variables.

In order to maximise \(\mathcal{L}\), we shall use variational derivatives.

\[
\delta \mathcal{L} = \frac{\langle \mathbf{u}(\mathbf{x}, 0) \rangle^2 (2\delta \mathbf{u}(\mathbf{x}, T) \mathbf{u}(\mathbf{x}, T)) - (2\delta \mathbf{u}(\mathbf{x}, 0) \mathbf{u}(\mathbf{x}, 0)) \langle \mathbf{u}(\mathbf{x}, T) \rangle^2}{\langle \mathbf{u}(\mathbf{x}, 0) \rangle^2}
- \int_0^T \langle \delta \nu, \left[ \frac{\partial \mathbf{u}}{\partial t} + 2(1 - s^2) \frac{\partial \mathbf{u}}{\partial z} - 4su\hat{z} + \nabla p - \frac{2}{Re} \nabla^2 \mathbf{u} \right] \rangle dt
- \int_0^T \langle \nu, \delta \left[ \frac{\partial \mathbf{u}}{\partial t} + 2(1 - s^2) \frac{\partial \mathbf{u}}{\partial z} - 4su\hat{z} + \nabla p - \frac{2}{Re} \nabla^2 \mathbf{u} \right] \rangle dt
- \int_0^T \langle \delta \Pi, \nabla \cdot \mathbf{u} \rangle dt
- \int_0^T \langle \Pi, \delta \nabla \cdot \mathbf{u} \rangle dt
- \int_0^T \langle \delta \Gamma(t), \mathbf{u} \cdot \hat{z} \rangle dt
- \int_0^T \langle \Gamma(t), \delta \mathbf{u} \cdot \hat{z} \rangle dt.
\] (5.3.58)
Rather than exhaustively derive all of the separate terms, we only include two examples here for illustration. For simplicity they are considered in Cartesian geometry with vector \( \mathbf{x} \) having components \( x_1, x_2 \) and \( x_3 \).

\[
\langle \nu, \delta \nabla^2 \mathbf{u} \rangle = \langle \nu_j \partial_i \delta u_j \rangle \quad (5.3.59)
\]

\begin{align*}
&= \langle \partial_i (\nu_j \partial_j \delta u_j) \rangle - \langle (\partial_i \nu_j) (\partial_j \delta u_j) \rangle \quad (5.3.60) \\
&= \int \nu_j \partial_i \delta u_j dS - \langle (\partial_i \nu_j) (\partial_j \delta u_j) \rangle \quad (5.3.61) \\
&= -\langle \partial_i ([\partial_j \nu_j] \delta u_j) + (\delta u_j \partial^2_i \nu_j) \rangle \quad (5.3.62) \\
&= -\int \delta u_j \partial_j \nu_j dS + \langle \delta \mathbf{u}, \nabla^2 \nu \rangle \quad (5.3.63) \\
&= \langle \delta \mathbf{u}, \nabla^2 \nu \rangle \quad (5.3.64)
\end{align*}

where we have made use of the product rule, the divergence theorem and boundary conditions. The pressure term requires an extra subtlety, as seen before, where we rewrite \( p = \lambda(t)z + \hat{p} \), with \( \hat{p} \) being strictly periodic.

\[
\langle \nu \delta \nabla \mathbf{p} \rangle = \langle \nu \delta \nabla [\lambda(t)z + \hat{p}] \rangle \quad (5.3.65)
\]

\[
= \langle \nabla \cdot (\nu \delta \hat{p}) \rangle - \langle \delta \hat{p} \nabla \cdot \nu \rangle + \langle \nu \cdot \delta \lambda(t) \hat{z} \rangle \quad (5.3.66)
\]

\[
= \int \nu \delta \hat{p} dS - \langle \delta \hat{p} \nabla \cdot \nu \rangle + \langle \nu \cdot \delta \lambda(t) \hat{z} \rangle \quad (5.3.67)
\]

\[
= -\langle \delta \hat{p} \nabla \cdot \nu \rangle + \delta \lambda(t) \langle \nu \cdot \hat{z} \rangle. \quad (5.3.68)
\]

The full derivation leads to the final formulation

\[
\delta \mathcal{L} = \langle \delta \mathbf{u}(x, T) \cdot \left[ \frac{2\mathbf{u}(x, T)}{(\mathbf{u}(x, 0))} \right] - \nu(x, T) \rangle \\
+ \langle \delta \mathbf{u}(x, 0) \cdot \left[ -\frac{2\mathbf{u}(x, 0)}{(\mathbf{u}(x, 0))} + \nu(x, 0) \right] \rangle \\
- \int_0^T \langle \delta \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} + 2(1 - s^2) \frac{\partial \mathbf{u}}{\partial z} - 4su\hat{z} + \nabla p - \frac{2}{Re} \nabla^2 \mathbf{u} \rangle \, dt \\
+ \int_0^T \langle \delta \mathbf{u} \cdot \frac{\partial \nu}{\partial t} + 2(1 - s^2) \frac{\partial \nu}{\partial z} + 4sv_3\hat{z} + \nabla \Pi + \frac{2}{Re} \nabla^2 \nu - \Gamma(t) \hat{z} \rangle \, dt \\
+ \int_0^T \langle \delta \hat{p} \nabla \cdot \nu \rangle \, dt
\]
$- \int_0^T \delta \lambda(t) \langle \nu \cdot \hat{z} \rangle dt$

$- \int_0^T \langle \delta \Pi \nabla \cdot u \rangle dt$

$- \int_0^T \delta \Gamma \langle u \cdot \hat{z} \rangle dt.$  \hspace{1cm} (5.3.69)

The meaning of this rather complicated equation is somewhat opaque. The problem we are tackling is one of maximisation. Extrema of the functional $\mathcal{L}$ are obtained when $\delta \mathcal{L} \equiv 0$. The six time integrals in the definition of $\delta \mathcal{L}$ are derived from Lagrangian multipliers and as such should evaluate to zero. Consequently we are looking to find when

$$\frac{\delta \mathcal{L}}{\delta u(x,0)} \equiv -\frac{2u(x,0)\langle u(x, T)^2 \rangle}{\langle u(x, 0)^2 \rangle^2} + \nu(x, 0) = 0 \hspace{1cm} (5.3.70)$$

and

$$\frac{\delta \mathcal{L}}{\delta u(x,T)} \equiv \frac{2u(x, T)}{\langle u(x, 0)^2 \rangle} - \nu(x, T) = 0. \hspace{1cm} (5.3.71)$$

The variational derivative contains two evolution equations which must be satisfied for all times. The first is the standard equation for the velocity field. The second is the equation for how $\nu$ evolves. The diffusion term in this second equation is of opposite sign to the velocity equation, implying that it can only be computationally integrated backwards in time. For this reason we refer to the equations for the original variables as the forward equations, and those for the adjoint variables as the backward equations.

This suggests an iterative scheme. By iteration $n$ we have initial velocity field $u^n(x, 0)$. We integrate this in time using the forward equations until time $T$. We choose $\nu^n(x, T)$ in order to satisfy equation 5.3.71. This is then integrated using the backward equation back to time $t = 0$. From $u^n(x, 0), u^n(x, T)$ and $\nu^n(x, 0)$ we can calculate $\delta \mathcal{L} / \delta u^n(x, 0)$. 

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This gradient is then used to start the next iteration as

\[
u^{n+1}(x, 0) = u^n(x, 0) + \epsilon \frac{\delta \mathcal{L}}{\delta u^n(x, 0)},
\]

(5.3.72)

\(\epsilon\) being the step-size, chosen to be small. We iterate until \(\delta \mathcal{L}/\delta u^n(x, 0)\) is sufficiently small, at which point a maximum has been found. The process can be started with any initial velocity field \(u^0(x, 0)\). The iterative method is schematically shown in figure 5.2.

One of the benefits of formulating the problem this way is that the only code it needs is a method to integrate the forward equation and a method to integrate the backward equation. The great similarity of these two equations means that the code described in section 5.1 only needs to be minimally modified to solve the backwards equations as well.

### 5.3.3 Nonlinear Extension

The formulation so far has been based around a linearised problem. Formally, linear fluid dynamics describes the behaviour of infinitely small disturbances. In this literal sense they cannot apply to the true scenario. It is, however, much simpler and computationally much cheaper to consider the linear problem. One then hopes that including the full nonlinear terms will not alter the result too much.
From a theoretical standpoint there is nothing to stop one including the nonlinear terms. In Cartesian coordinates, the velocity evolution term in equation 5.3.57 needs to now include $\nu(u \cdot \nabla u)$, which is then followed through the working as shown below.

\[
\langle \nu \cdot \delta (u \cdot \nabla u) \rangle = \langle \nu \cdot (\delta u \cdot \nabla)u \rangle + \langle \nu \cdot (u \cdot \nabla)\delta u \rangle = \langle \nu_i \delta u_j \partial_j u_i \rangle + \langle \nu_j \delta u_i \partial_i u_j \rangle \tag{5.3.73}
\]

\[
\begin{align*}
= \langle \nu_i \partial_j (\delta u_j u_i) \rangle - \langle \nu_i u_i \partial_j \delta u_j \rangle + \langle \delta u_j u_i \partial_j \nu_i \rangle - \langle \delta u_j \partial_i u_i \delta \nu_j \rangle \tag{5.3.74}
\end{align*}
\]

\[
= \int (\nu \cdot u) \delta u \cdot dS - \langle u \cdot (\delta u \cdot \nabla)\nu \rangle \tag{5.3.75}
\]

\[
= \int (\nu \cdot \delta u) u \cdot dS - \langle \delta u \cdot (u \cdot \nabla)\nu \rangle \tag{5.3.76}
\]

\[
= -\langle \delta u \partial_i (\nu_i \delta u_j u_i) \rangle - \langle \delta u \cdot (u \cdot \nabla)\nu \rangle \tag{5.3.77}
\]

In cylindrical coordinates the algebra is a little more involved, and the final term slightly more complex.

\[
= -\langle \delta u \left[ \left( u_i \partial_j \nu_i \right) + \frac{1}{s} (\nu_\phi u_s - \nu_s u_\phi) \partial_\phi \right] \rangle - \langle \delta u \cdot (u \cdot \nabla)\nu \rangle, \tag{5.3.79}
\]

where $\partial_\phi = (\partial_s, \frac{1}{s} \partial_\phi, \partial_z)$.

Although it is straightforward enough to write down the equations, we quickly see that the calculation involved presents a difficulty in that at each time step in the backward equation, the corresponding velocity field from the forward equation is needed. To store all of this information can quickly become prohibitively expensive and we instead will need a work-around. This is done by a method known as check pointing. In this method the forward velocity field is only saved every, say, 1000 time
Table 5.2: A comparison of linear transient growth results of the code presented here and the results published by Meseguer and Trefethen (2003). Each result is for a single azimuthal wavenumber \((m = 1)\) and a single streamwise mode corresponding to a wavenumber of \(\alpha\). Our results correspond to a spatial resolution of 35 modes radially and a time step of \(dt = 5 \times 10^{-4} D/U\).

<table>
<thead>
<tr>
<th>Re</th>
<th>(\alpha = 0)</th>
<th>(\alpha = 1)</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>Meseguer &amp; Trefethen</td>
<td>Here</td>
</tr>
<tr>
<td>(10^2)</td>
<td>1.066</td>
<td>1.065</td>
</tr>
<tr>
<td>(10^3)</td>
<td>8.510</td>
<td>8.509</td>
</tr>
<tr>
<td>(10^4)</td>
<td>84.92</td>
<td>84.91</td>
</tr>
</tbody>
</table>

steps. Then when calculating the backwards evolution we break the time line into 1000 time step chunks. For each of these, the forward simulation is performed again from the relevant starting point. Thus for a \(\sim 50\%\) increase in the calculation time, we no longer need to worry about memory constraints.

5.4 Testing the Code (Part 2)

5.4.1 Linear Results

Linear transient growth in pipe flow has been examined since the early nineties (Bergstrom, 1993; O’Sullivan and Breuer, 1994; Schmid and Henningson, 1994), but in order to test our code we referred to Meseguer and Trefethen (2003) who provide a very thorough analysis of the linear problem. One of the simpler problems they considered is transient growth of modes with unitary azimuthal wavenumber and either 0 or 1 as the streamwise wavenumber. We calculated the transient growth at \(Re = 100, 1000\) and 10000 for both these modes and compared our results to their’s. The close match is shown in table 5.2.
5.4.2 Nonlinear Results

There are no published results for nonlinear transient growth, so we need to consider an alternative way of testing the nonlinear parts of the code. The most straightforward way to test the code is to try to reproduce the linear results using very small amplitudes. This, however, is not very informative as any additional terms which are quadratic in the velocity field will have negligible effect in the limit of small amplitudes.

Instead we tackle the problem of heteroclinic connections. A heteroclinic connection is simply a path in phase space between any two different exact solutions. Formally it must begin on the unstable manifold of one solution and finish on the stable manifold of the second. This is numerically impossible, and so various approaches have been used in an attempt to identify them (Halcrow et al., 2009). Instead of looking for a true heteroclinic connection, we shall look for a path which simply begins ‘close’ to one solution and ends ‘close’ to another.

To formulate this is a similar manner to transient growth, we consider the functional

$$\mathcal{L} = \langle [u(x,T) - v_2(x)]^2 \rangle - \lambda \langle [u(x,0) - v_1(x) - E]^2 \rangle - L,$$  \hspace{1cm} (5.4.80)

where $L$ represents our usual constraints. Physically, minimising this functional corresponds to minimising the distance between the velocity field evolved to time $T$ and a second unspecified velocity field $v_2$. This is subject to the additional constraint that velocity field at time 0 is close to the velocity field $v_1$. If $v_1$ and $v_2$ are travelling waves, then this will give us what we shall term a pseudo-heteroclinic connection between these two solutions.

Taking the usual variational derivative of 5.4.80 gives

$$\delta \mathcal{L} = 2\langle \delta u(x,T)[u(x,T) - v_2(x)] \rangle - 2\lambda \langle \delta u(x,0)[u(x,0) - v_1(x)] \rangle \nonumber$$
$$\nonumber - \delta L \hspace{1cm} (5.4.81)$$

\hfill 120
\[
\langle \delta u(x, T)|[2(u(x, T) - v_2(x)) - \nu(x, T)]\rangle \\
+ \langle \delta u(x, 0)|[2(v_2(x) - u(x, 0)) + \nu(x, 0)]\rangle \\
- \delta L', \quad (5.4.82)
\]

where \(\delta L'\) is the derivative of the usual Lagrangian constraints except with the term \(\langle [\nu u]^T_{i=0} \rangle\) removed.

There is now an additional unknown in the form of \(\lambda\) which is included to maintain the energy separation of \(u\) and \(\nu\). It can be calculated explicitly from

\[
\langle [u^n(x, 0) - v_1(x)]^2 \rangle = \langle [u^{n+1}(x, 0) - v_1(x)]^2 \rangle \quad (5.4.83)
\]

when we note that

\[
u^{n+1} = u^n + \frac{\delta L'}{\delta u} \quad (5.4.84)
\]

\[
\Rightarrow \langle (u^n - v_1)^2 \rangle = \langle [u^n + \epsilon(2\lambda(v_1 - u^n) + \nu^n) - v_1]^2 \rangle \quad (5.4.85)
\]

\[
\Rightarrow 0 = 2\epsilon \langle (u^n - v_1)\nu \rangle + \epsilon^2 \langle (\nu^n)^2 \rangle \\
+ 4\epsilon \lambda \langle (u^n - v_1)^2 \rangle + 4\epsilon^2 \lambda \langle (u^n - v_1)\nu^n \rangle \\
+ 4\epsilon^2 \lambda^2 \langle (u^n - v_1)^2 \rangle \quad (5.4.86)
\]

This gives us a quadratic equation for \(\lambda\).

Now we must choose suitable solutions to look for a connection between. It seems most reasonable to look for two travelling waves which are already believed to be close to one another in phase space. To this end we chose S1 and M1 from earlier chapters which are known to join in a bifurcation. For an axial wavenumber of \(\alpha = 0.75\) this bifurcation takes place at \(Re = 1770\), and so we looked for a connection between these two solutions at \(Re = 2000\). We kept within a low truncation of only 9 azimuthal modes and 3 axial ones, but within this ‘artificial’
dynamical system everything is self-consistent. The target integration
time was fixed at $5D/U$ with a time step of $0.25 \times 10^{-3} D/U$. We choose
that the initial velocity field should satisfy

$$\frac{\langle (u(x, 0) - v_1)^2 \rangle}{\langle v_1^2 \rangle} = 2.5 \times 10^{-3}. \quad (5.4.87)$$

We let the program run for 1000 iterations during which time we
observed a decay in $\langle (\partial L / \partial u_0)^2 \rangle$ of around 8 orders of magnitude. The
residual after this time was still decaying but we ended the calculation
at this point due to time restrictions. The convergence is shown in figure
5.3.

In order to measure how close the flowfield is to a given solution
without returning to the energy norm we instead define two correlation
functions, $I_{tot}$ and $I_{uv}$, after Kerswell and Tutty (2007). These mea-
sure the correlation between any given velocity field, $u$, and a particular
travelling wave $u_{TW}$. We first define the modified velocity fields

$$v := u - u_{TW}$$ (5.4.88)

$$v_{TW} := u_{TW} - \bar{u}_{TW}.$$ (5.4.89)

The modified velocity fields have the mean flow of the travelling wave subtracted from them as otherwise this dominates any correlation calculation performed. From these we define functions

$$I_{tot} := \max_{\phi,z} \frac{\langle v, v_{TW} \rangle}{\sqrt{\langle v, v \rangle \langle v_{TW}, v_{TW} \rangle}}$$ (5.4.90)

$$I_{uv} := \max_{\phi,z} \frac{\langle v^\perp, v_{TW}^\perp \rangle}{\sqrt{\langle v^\perp, v^\perp \rangle \langle v_{TW}^\perp, v_{TW}^\perp \rangle}}$$ (5.4.91)

with $u^\perp = (u, v, 0)$. Thus $I_{tot}$ correlates the full velocity field while $I_{uv}$ only compares the cross-stream component. This is important as typically the cross-stream velocity field will be an order of magnitude smaller than the axial flow field. In Kerswell and Tutty’s paper they define close approaches to these travelling waves in terms of a parameter $\lambda$ by

$$I_{tot} > \lambda$$ (5.4.92)

$$I_{tot} + I_{uv} > 2\lambda.$$ (5.4.93)

If $\lambda$ was greater than 0.6, then the flowfield was deemed a close visit. As we are interested in the proximity to two different travelling waves we use this to define $\lambda_{S1}$ and $\lambda_{M1}$. A plot of these is shown in figure 5.4, where the initial state (marked with a circle) is allowed to evolve until it eventually decays away. Initially the value for $\lambda_{S1} = 0.89$, while the closest approach to M1 corresponds to $\lambda_{M1} = 0.98$. These quantities certainly suggest close approaches of the flow, and even if it is not a true heteroclinic connection, it is remarkable nonetheless.
Figure 5.4: The solution found after 1000 iterations is allowed to evolve for long time until it decays to the laminar state. The thicker part of the line shows the part of the evolution considered in the optimisation procedure. The initial velocity field equates to the point denoted with a circle and then after the target optimisation time ($5D/U$) has elapsed it has reached the square. If allowed to continue, the flow continues to a point even closer to M1, before smoothly decaying to the laminar state. The insets show the instantaneous velocity fields at $t = 0$ and $t = 5D/U$. 
Chapter 6

Nonlinear Transient Growth

6.1 Introduction

In the previous chapter we introduced the idea of transient growth. In the linear setting it has been well studied both in pipe flow and beyond. The variational method derived is flexible enough in methodology to be applied not only to the canonical shear flows, but also to more complex geometries such as the flow over a backward-facing step (Blackburn et al., 2008), or to flows with more complex governing equations such as in a thermoacoustic setting (Balasubramanian and Sujith, 2008).

In the case of pipe flow the linear optimal is simply given by a pair of large streamwise rolls, each of which fills around half the pipe. This is illustrated in figure 6.1a. The reason this initial perturbation causes such strong growth is relatively intuitive. There is, of course, an underlying shear in the pipe with fast-moving fluid in the centre and slow fluid near the walls. The rolls drag slow fluid from the edge into the middle while at the same time they pull fast fluid from the centre out to the edge. This creates anomalies in the streamwise velocity field (streaks) which
Figure 6.1: a) The perturbation in linearised pipe flow which produces the most growth. It consists of a two streamwise independent rolls each of which takes up around half the pipe. Thefd is no streamwise component. b) The streaky flow structure which the rolls create after the maximum amount of energy growth. Again they are streamwise independent and each streak takes up around half the pipe.

are much more energetic than the rolls which created them and can be seen in figure 6.1b. Simple comparison tells us that the rolls will decay over a timescale of $O(Re^{-1})$. Thus rolls of $O(\epsilon)$ lasting a period of $O(Re)$ produce streaks which are $O(\epsilon Re)$. The velocity field produce is $O(Re)$ larger than the perturbation and so the energy growth is expected to be $O(Re^2)$. This is borne out by the calculations of Meseguer and Trefethen (2003) over four orders of magnitude of Reynolds number.

We introduce the following notation. The time which corresponds to the maximum amount of growth in the linear problem is $T$. The initial velocity field which produces this growth is $U_0$ (normalised such that it has unit energy) and the velocity field it produces at the point of maximum growth is $U_T$. We will give energies in units of the energy of the laminar flow in a pipe $\frac{1}{2}\pi D$ in length.

The simplicity of this result is very pleasing and has led to it being viewed as a key mechanism, whereby turbulence can be triggered by much smaller amplitude pertubations. There are, however, a few draw- backs with this idea. Firstly, the result is a linear result. This may seem
like stating the obvious, but is vitally important. In the strictest sense, it only applies to infinitessimal perturbations. An $O(Re^2)$ growth may seem like a lot, but an infinitessimal perturbation after a large amount of growth is still infinitessimal. The counter argument to this runs that it will be a reasonable approximation at low amplitudes, and given the size of the growth only a small amplitude perturbation is required. This argument is further strengthened by recognising that the only source of energy for the perturbation is the underlying shear, and this only enters the equations for the perturbations development through the linear terms. More rigourously let us view the flow as a base flow plus a perturbation, $U + u$. The disturbance energy is then given by

$$E = \frac{1}{2} \int |u|^2 dV.$$  \hfill (6.1.1)

We can manipulate the full (nonlinear) Navier-Stokes equations by taking the scalar product of it with $u$. Integrating this and applying incompressibility and periodic boundary conditions leads us to the Reynolds-Orr equation,

$$\frac{\partial E}{\partial t} = -\int (u \cdot \nabla)U \cdot u dV - \frac{1}{Re} \int |\nabla u|^2 dV.$$  \hfill (6.1.2)

The nonlinear term does not appear in this, and so the total gain in the disturbances energy must be a linear process - a balance between mean flow interaction and viscous dissipation.

A second problem is that the perturbation is streamwise independent and it can be demonstrated that such a perturbation cannot trigger turbulence. We would therefore require the addition of some noise, presumably at the peak of the growth. Finally, we note that both turbulence and transition are fundamentally nonlinear phenomena. It is therefore questionable how relevant a linear study can be. In this chapter we seek to address this by performing the full nonlinear transient growth
calculation.

6.2 The Problem

With the inclusion of nonlinearity in the code we introduce the added complication of initial amplitude dependency. For vanishingly small initial conditions we expect to recover the linear result. At the more energetic end of the scale we have turbulence to consider. In order to investigate the effect of the initial amplitude on the optimal solution we reformulate the problem slightly as

\[
\mathcal{L} = \langle u(x, T)^2 \rangle - \lambda \left[ \frac{1}{2} u(x, 0)^2 - E_0 \right] - L. \tag{6.2.3}
\]

With this formulation we can set the initial energy of the disturbance as being \(E_0\) and look for the perturbation of this size, which produces the most growth. Formulating the problem this way alters the end conditions to be

\[
\frac{\delta \mathcal{L}}{\delta u(x, T)} \equiv 2u(x, T) - \nu(x, T) = 0 \tag{6.2.4}
\]

and

\[
\frac{\delta \mathcal{L}}{\delta u(x, 0)} \equiv -2\lambda u(x, 0) + \nu(x, 0) = 0. \tag{6.2.5}
\]

The Lagrange multiplier \(\lambda\) is found by insisting that the amplitude of the initial disturbance is unchanged between iterations so that

\[
\langle u^n(x, 0)^2 \rangle = \langle u^{n+1}(x, 0)^2 \rangle \tag{6.2.6}
\]

\[
= \left( u^n(x, 0) + \epsilon [\nu(x, 0) - 2\lambda u(x, 0)] \right)^2 \tag{6.2.7}
\]

giving a quadratic equation for \(\lambda\). \(\epsilon\) is the stepsize taken.
6.3 Preliminary Studies

6.3.1 Streamwise Independent Perturbations

For vanishingly small perturbations the linearised Navier-Stokes equations are mathematically correct, while for small perturbations we expect them to give an accurate approximation. The question becomes, at what energy does linearising the problem cease to provide relevant results. To tackle this we consider the problem of nonlinear transient growth in a streamwise independent environment. This has been considered before, at least in the case of a boundary layer (Zuccher et al., 2006). Here they discovered that nonlinearity did not notably alter the optimal disturbance, and we expect a similar result in pipe flow. Their results are shown in figure 6.2. Increasing the energy of the perturbation has very little impact on the form of the optimal solution. To the naked eye the difference is almost imperceptible, even when the initial energy is increased by two orders of magnitude.

Returning to pipe flow, we now consider the effect of amplitude on the streamwise independent optimal. Firstly, we will perform the nonlinear optimisation procedure setting $\alpha = 0$ and varying $E_0$. This will capture
Figure 6.3: How the growth exhibited by the streamwise independent nonlinear optimal dies off with increasing amplitude. The calculation is for $Re = 1750$. The black dotted line is the linear result, which the nonlinear calculation (blue crosses) asymptotes towards for small amplitudes. The solid blue line is the growth observed when a nonlinear simulation is seeded with a scaled version of the linear optimal. The energy is given in units of the laminar flow in a pipe which is $\frac{1}{4}\pi D$ long. The resolution corresponds to 25 radial modes and 14 azimuthal modes, with $dt = 3.6 \times 10^{-3}(D/U)$. For all calculations the growth is given at fixed final time $t = T$, the optimal time scale in the linear case.
any changes in the form of the optimal, however small they might appear. Secondly we can take the linear optimal and simply use it to seed a nonlinear simulation after scaling it to the correct initial energy.

We expect these two methods to agree well for small amplitude perturbations, and indeed they must in the limit of vanishingly small perturbations. As we increase the amplitude, we can expect a small discrepancy as the flow field begins to be modified, at first mildly, by the nonlinearities. For very large $E_0$ the two methods may give strongly differing results.

We plot the results of both methods in figure 6.3. In this plot the dashed line represents the linear result, independent of amplitude. The blue line is the growth given by seeding a nonlinear simulation with the velocity field

$$u = \sqrt{2E_0} U_0.$$  \hspace{1cm} (6.3.8)

The crosses correspond to points where we have explicitly calculated the nonlinear optimal. As expected the linear result, the growth of the rescaled linear optimal and the nonlinear optimal all coincide for small amplitudes. However as $E_0$ is increased the amount of growth produced begins to decrease after $E_0 \approx 10^{-6}$. The two methods for determining the effect of nonlinearity continue to give good agreement beyond this point, suggesting that form of the nonlinear streamwise independent optimal is nearly unchanged from the linear result, $U_0$. It is not until $E_0 \approx 5 \times 10^{-4}$ that the two methods begin to diverge and there is no significant difference until $E_0 > 10^{-3}$. In figure 6.4 we see how the appearance of the optimal mode varies with $E_0$.

6.3.2 Weakly Streamwise Dependent Perturbations

In some sense the iterative method can be thought of as being the same as any other dynamical system. Instead of evolving through the continuous parameter of time, $t$, we now have the discrete parameter of iterations,
Figure 6.4: From left to right: The streamwise independent optimal solution for $E_0 = 10^{-7}, 10^{-4}$ and $2 \times 10^{-3}$. For lower energies the form of the optimal perturbation doesn’t vary notably, even as the growth exhibited begins to diverge from the linear result. Only at a larger amplitude is the form of the optimal altered. The first two perturbations have no axial component. The third does, but it has been suppressed for easier comparison.

$n$. The formulation of the dynamical system is given by

$$\frac{\partial}{\partial n} u_0(x, n) = F\{u_0(x, n)\},$$

(6.3.9)

where $F\{u_0(x, n)\} = \nu_0(x, n) - 2\lambda u_0(x, n)$ and $\nu$ and $\lambda$ are themselves functions of $u_0(x, n)$.

If considered in this manner, then we can see that optimal perturbations correspond to fixed points, and we can expect them to have their own stabilities. For low energies the streamwise independent optimal is a (global) maximum and so is entirely stable. Increasing the energy is likely to have little effect at first unless a critical threshold is reached at which point one of two things can happen. Either an unstable direction will appear and the maximum becomes a saddle point, or an entirely unconnected new maximum appears leaving the streamwise independent maximum as a local optimum. It is difficult to disprove that the second option has occurred, but we attempt to combat it by using three different initial velocity fields in the iterative process. These are a ‘noisy’ version of the linear result, the asymmetric travelling wave and a turbulent velocity field - each scaled to possess the desired amount of
initial energy. Our results were independent of the seeding velocity field and so we henceforth assume that the former possibility is the correct one.

The previous section demonstrated that this critical threshold doesn’t exist in the streamwise independent subspace, and so any such threshold must be streamwise dependent. Close to the threshold, if we expand the streamwise direction in Fourier modes, we expect the energy in the $n$th Fourier mode to be $O(\epsilon^n)$. We can therefore perform a weakly nonlinear study by just considering the first streamwise Fourier mode which will accurately capture the behaviour close to this threshold.

We performed the weakly nonlinear calculation for $Re = 1750$ and $\alpha = 2$ (a $\frac{1}{2}\pi D$ pipe). For this calculation we kept the final time fixed as $T$, the time scale of the linear optimal ($T = 42.68D/U$ for $Re = 1750$). Our resolution corresponded to 29 modes azimuthally and 25 radially, while just 3 modes axially (the streamwise independent mode as well as $\pm \alpha$). The initial velocity field that was used to seed the calculations was taken from a turbulent run (but scaled to give the correct $E_0$). Figure 6.3 would lead us to suppose that nonlinear effects will become important where the amount of growth of the two dimensional optimal departs the linear result.

To investigate the formulation of equation (6.3.9) we performed a nonlinear optimisation for a range of initial energies. In figure 6.5 we plot the variation of $F$ with iterations. As with any other dynamical system we have exponential decay towards the stationary point, with the gradient given by the ‘least stable’ direction. For low $E_0$ the least stable direction is given by the second most optimal mode of the linear problem ($m_0 = 2$). However as the energy is increased a new least stable mode appears, and with increasing energy it becomes less stable. The precise value at which it becomes neutral is much more difficult to resolve than the normal flow dynamics. Although the rest of the calculations here were performed with $dt = 3.6 \times 10^{-3}D/U$, to fully resolve the point
Figure 6.5: **Outer:** Convergence of the weakly nonlinear calculation at $Re = 1750$ in a $\frac{1}{2}\pi D$ pipe with $dt = 3.6 \times 10^{-3} D/U$. Each line corresponds to an increasing $E_0$ from bottom to top of $4 \times 10^{-6}, 6 \times 10^{-6}, 8 \times 10^{-6}$ and $10 \times 10^{-6}$. The lowest energy converges smoothly to a fixed point ($F = 0$). The next two have a new least stable direction compared to the first, but still converge. For $E_0 = 10 \times 10^{-6}$ the new direction is close to neutral. **Inner:** Convergence of the weakly nonlinear calculation at $Re = 1750$ in a $\frac{1}{2}\pi D$ pipe with $dt = 5 \times 10^{-4} D/U$. Although $dt = 3.6 \times 10^{-3} D/U$ is small enough to resolve all the generic dynamics of this system, it does not fully resolve the exact point at which the system becomes unstable. The fully resolved plot with $dt = 5 \times 10^{-4}$ is shown here. The lines correspond to $E_0 = 6 \times 10^{-6}, 8 \times 10^{-6}, 10 \times 10^{-6}, 12 \times 10^{-6}$ and $14 \times 10^{-6}$. 
of instability we have to resort to \( dt = 5 \times 10^{-4}D/U \). The contrast between the two is shown between the outer and inner plots of figure 6.5.

### 6.4 Nonlinear Optimal Perturbations

#### 6.4.1 Parameter Selection

The weakly nonlinear study highlights the region after which nonlinearity affects the full problem to the extent that the optimal solution is no longer a minute modification of the linear optimal. Including the full nonlinearity is rather more difficult than the previous and there are two main obstacles. The first is time. The problem is quadratic in the axial resolution - increasing the number of axial modes from being \(-\alpha, \ldots, +\alpha\) to \(-5\alpha, \ldots, +5\alpha\) corresponds to almost four times as many modes (11 rather than 3). Each mode’s evolution has to be calculated which would give a linear increase in run time, however the nonlinear term for each mode is a calculation linear in the number of modes. This gives a potentially much larger run time.

The second difficulty that we face is that if too large a value of \( E_0 \) is used, then it is possible to trigger, at least transient, turbulence. This leads to the problem no longer being smooth and we cannot expect convergence. This, then, is the importance of the weakly nonlinear study. It allows us to quickly assess where it is both useful and possible to perform the full nonlinear analysis.

Forearmed with the knowledge from the previous section, we performed the nonlinear computation for \( Re = 1750, \alpha = 2 \) and \( E_0 = 2 \times 10^{-5} \). We retained the same azimuthal and radial resolutions as before, but now additionally considered 11 axial modes. We seeded the iterations with the 2D optimal, plus a small quantity of noise.
Figure 6.6: $Re = 1750, \alpha = 2, E_0 = 2 \times 10^{-5}$. The iterations are seeded with a noisy version of the 2D optimal. After approaching two saddle points, it eventually converges onto the nonlinear optimal.

### 6.4.2 The Nonlinear Optimal

The evolution of the iterative procedure is shown in figure 6.6. For the first 30 or so iterations the procedure remains close to the original 2D optimal. Slowly it drifts away along the unstable direction, as the 2D optimal is a saddle point rather than a maximum. It then approaches a second saddle point before again drifting away. It is eventually attracted to the new maximum, which corresponds to a growth of 330.2, compared to the linear result of 221 for the 2D optimal at this $E_0$, which exhibits a growth of 215.

In figure 6.7 we compare the time evolution of the linear and nonlinear optimals, as well as their physical appearance. We see that they are quite different. The nonlinear perturbation is localised azimuthally and has an almost critical layer type structure. With time, however, it smooths out into a simpler form of two slow streaks and one fast, qualitatively
Figure 6.7: The time evolution of the linear and nonlinear optimals. The nonlinear optimal reaches a maximum earlier than the linear optimal, at $t_{\text{max}} = 16.6$. On the right is the initial form of the optimal perturbation (top) and the form of disturbance at $t_{\text{max}}$.

similar to the asymmetric travelling wave, but this time the position of the fast and slow streaks have switched. Successive slices across the pipe during the nonlinear optimals evolution are shown in figure 6.8. The complexity of its evolution is clear.

6.4.3 Variation with $E_0$

We now turn to the question, how does the optimal perturbation vary with $E_0$? The most computationally efficient way to investigate this is to simply rescale the optimal to other energies and see how much growth it provides. Certainly in the case of the 2D optimal we saw that this was reasonably accurate. In figure 6.9 we do just this. The red line represents the rescaled linear optimal, while the blue line is the rescaled nonlinear optimal. As expected, for low energies the linear result is optimal. As we increase the energy, the nonlinear result grows and the linear result subsides until the two cross over at $E_0 \approx 10^{-5}$. Increase the energy much further and the nonlinear results grow chaotic.
Figure 6.8: Instantaneous slices through the pipe showing the optimal perturbations evolution in time. Slices are at \( t = 0.0, 2.5, 5.0, 7.5, 10 \) and 12.5 (left to right, top to bottom). For \( 12.5 \leq t \leq t_{\text{max}} \) the form of the perturbation does not alter much, the streaks simply grow in intensity.
The cyan lines represent the average and upper/lower turbulent energy bounds. We note that they match up well with the chaotic regime of the nonlinear optimal, which we interpret as the optimal being able to trigger turbulence - that is to say it has crossed the edge of chaos. The final line on the graph (green) is the saddle through which the optimisation routine passes on its way from the 2D optimal to the nonlinear optimal. This is never a maximaum, but appears to instead only exist as a saddle point.

In the inner plot of this figure, we compare the scaled optima’s growth with the growth obtained by the full nonlinear computation (the crosses). As must necessarily be the case, the crosses are always higher than the lines, but they are reasonably well-matched.

6.5 Discussion

The great question we have attempted to tackle in this chapter is how does a small perturbation grow into a much more energetic transient? For some time the view has been that the best method is via transient growth, and that the linear result of two large rolls is the most efficient way to approach high energy turbulence. In this chapter we have first demonstrated that at even quite moderate energies the amount of growth two rolls produce is greatly inhibited. The large growth expected is not realised well before (in terms of amplitude) turbulence can be observed.

Instead we have suggested that as the linear result declines in efficiency, a new, more productive perturbation appears. This perturbation is grossly different in form, bearing little resemblance to the original disturbance. Unlike the linear optimal, the new perturbation is fully capable of triggering a turbulent episode. We can show this by taking the initial condition corresponding to the circle in figure 6.9 and allowing it to evolve for larger times. This evolution is shown in figure 6.10 where it clearly undergoes a turbulent episode.
Figure 6.9: **Outer:** the amount of growth produced by scaling the linear optimal (red line) and the nonlinear optimal for $E_0 = 2 \times 10^{-5}$ (blue line). The green line is the ‘second saddle’, which the iteration procedure passes through on the way to the nonlinear optimal. The black lines are the average turbulent energy (dashed) and the upper and lower turbulent energy bounds (dash-dot), all from numerical runs. **Inner:** Close-up of the outer plot. The crosses mark the results of full nonlinear optimisation calculations. A reasonable match is observed.
Figure 6.10: We seed a nonlinear simulation with the nonlinear optimal calculated in this chapter. The simulation is performed at the same Reynolds number (1750) and the same domain size ($\alpha = 2.0$) as the optimisation. The initial energy of the perturbation, however, has been scaled from $2 \times 10^{-5}$ to $6.3 \times 10^{-5}$. The perturbation undergoes a turbulent episode for around $200D/U$ before relaminarising.
We end with the conclusion that there is a large gap between the size of perturbation the linear result can trigger, and the size of perturbation required to trigger turbulence. This gap can be filled by the nonlinear optimal discussed here, which uses a completely different mechanism to that of the linear result.

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