Mechanics of materials (solid mechanics), MENG11100, in 94 solved problems

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ABSTRACT

An absolute minimum of solid mechanics knowledge required to study more advanced concepts in years 2 and above. The topics include: stress and strain tensors, stress equilibrium, solutions of simple 1D and 2D stress problems, principal values and directions, tensor rotations, maximum shear orientation and value, Mohr’s circle, linear isotropic elasticity, idea of a solid mechanics problem, uniaxial stress/strain states, statically indeterminate systems, slender beam bending theory, properties of cross sections, plane strain/stress, axisymmetry, torsion, elastic stability and buckling. Material and structural nonlinearity, anisotropy, inelasticity, general solution methods to elastic problems are excluded.

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### Table of Contents

1. Abbreviations .................................................. 1
2. Notation ......................................................... 1
3. Recommended reading ........................................... 1
4. One-dimensional stress/strain analysis ........................ 3
   4.1. Fundamental ideas of stress and strain and linear elasticity .......... 3
   4.1.1. Stress ............................................. 3
   4.1.2. Strain ............................................ 6
   4.1.3. Young’s modulus ..................................... 7
   4.1.4. Poisson’s ratio ...................................... 7
   4.1.5. Displacement ......................................... 9
   4.1.6. Elastic energy ....................................... 9
   4.2. Pin-joined frame ......................................... 9
   4.3. Statically indeterminate systems .................................. 11
   4.4. Euler-Bernoulli (slender beam) bending theory .................... 12
      4.4.1. Properties of cross sections .......................... 16
      4.4.2. Plasticity in bending ................................ 18
   4.5. Stability .............................................. 18
      4.5.1. Buckling of columns ................................. 20
5. Three-dimensional stress/strain analysis ....................... 22
   5.1. Motivation .............................................. 22
   5.2. Stress ................................................. 24
      5.2.1. Elementary cube of material .......................... 27
      5.2.2. Conservation of linear momentum ..................... 27
      5.2.3. Conservation of angular momentum .................... 28
   5.3. Tensors ................................................. 29
      5.3.1. Eigenvalue / vector ................................ 30
   5.4. Strain ................................................. 31
   5.5. Maximum shear value ..................................... 33
   5.6. Mohr’s diagram ......................................... 33
   5.7. Elasticity .............................................. 34
   5.8. Solving solid mechanics problems ................................ 38
6. Special cases .................................................... 38
   6.1. Two-dimensional stress/strain problems ........................ 39
      6.1.1. Plane stress ....................................... 39
      6.1.2. Plane strain ....................................... 40
      6.1.3. Axisymmetric ...................................... 41
      6.1.4. Torsion ........................................... 42
   6.2. Application of tensor theory to properties of areas ................ 45
7. Example problems ................................................ 46
8. Solutions to example problems ................................... 51
References


Brillouin, 1964.


Calcote, 1968.

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Hosford, 2005.

Hosford, 2010.
Hunter, 1976.

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Mase, 1970.


Scipio, 1967.


Sokolnikoff, 1956.

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Tadmor, 2012.

Valliappan, 1981.
Index

A
acceleration 3, 81-82, 84
alternating tensor 30, 85-87
area
  centroid 17
axysymmetric 42
  geometry 44

B
basis vectors 88
  orthonormality 90, 95-96
bending 12, 116
  cross section (see moments of area)
  curvature 14
  deflection 12, 116
  distributed load 13, 15
  Euler-Bernoulli theory 12
fibres 12
four-point 15
gradient 15
moment 18, 116
neutral line 12, 18
plasticity 18
pure 15
radius of curvature 14
second moment of area 14
shear force 15
slender beam 12
thick beams 41
body force (see force)
boundary conditions 38, 81-82
buckling 20
column 20
critical state 20

C
compatibility
  displacement 57, 59
 compliance tensor
    major symmetry 106
 compliance
tensor 106
compression modulus (see elastic bulk modulus)
configuration
  current 31
  deformed (see current)
  original 31
  undeformed (see original)
conservation
  angular momentum 28
  linear momentum 3, 24
constitutive model 18, 35
constraint 41
continuum mechanics 32
coordinate system 3, 16, 22, 29, 39, 88, 96-97, 121
  left-handed 84
  orthonormal 29
  principal 90-91, 96
  right-handed 84, 88
  rotated 29, 91, 98
coordinate transformation 29, 34, 38-39, 85-87, 91-92, 96-97, 113, 121
  invariant 29
coordinate
  axes 16
  axis 6, 80
cross section
  2nd moment 116
  polar moment 116

D
deformation 31
deformation (see strain)
deformation gradient 32, 92-93, 96-97
deformation
  reversible 32
displacement 6, 9, 31, 94
  field 94
  gradient 7, 32, 92-94, 113
  vector 31, 38, 92, 112

E
eigenvalues (see rank 2 tensor principal values)
eigenvectors (see rank 2 tensor principal vectors)
Einstein’s summation convention 26
elastic tensor
  major symmetry 105-106
  minor symmetry 105
elastic
  anisotropic 37
  behaviour 35, 38
  bulk modulus 38, 108
  compliance 36
  compliance tensor 108-109
  conservative 36
  energy density 106
  inverse 36
  isotropic 37
Lamé constants 37, 41, 43, 107, 110, 115
linear 36-37
lossless 36
non-linear 36
path independent 36
potential 36
recoverable 36
reversible 36
shear modulus 38, 107
superposition 38, 118
tensor 35-37, 105-106, 108, 110
Young’s modulus 40, 109-110, 115
elasticity 6, 18
energy 9
linear 7, 115
one-dimensional 6
Young’s modulus 7
elementary cube of material 27, 74, 91, 102, 119
elongation 6, 106
energy
elastic 9
equilibrium
critical 20
force 3
moment 3, 20
new 20
perturbation 19
stable 19-20
unstable 19-20

F
force 3, 82-83, 88
body 26, 42, 82
distributed 83
gravity 80
internal 24
reaction 54
free body diagram 3, 81-83

G
Green’s theorem 114

I
impulse 3
index
dummy 26, 75
live 75
instability (see stability)

K
kinematics 31
Kronecker delta 29, 32, 76, 80, 87, 92, 110
Kronecker delta (see also rank 2 tensor identity)

L
Lagrange multiplier 99
linear

M
material
constitutive model 38
elastic 37
incompressible 108
isotropic 107
symmetry 37
matrix
determinant 85
symmetric 86
Mohr’s diagram 33, 103-105, 120
moment 3
moment of area
2nd 121
first 121
tensor 121
moments of area
first 16
polar 17
second 17
momentum
angular 45
linear 3
motion 31, 93

N
Newton’s laws
second 3
third 4

O
operator
divergence 75, 110
gradient 74, 110
optimisation
constrained 99
objective function 99

P
pin-joined frame 9
plastic limit 18
plastic
flow 18
plasticity 18
Poisson’s ratio 7-8, 38, 40, 106-107, 115
0.5 108
position vector
derivative 80
pressure 114
profile
  displacement 6
  force 4

rigid body
  rotation 57
rotation tensor (see rank 2 tensor
  rotation

S
sectioning method 3
shear
  maximum value 33
  pure 95
  simple (see pure)
shoulder 3
solid mechanics 38
  problem 39
solution 39
stability 18
  elastic 20
Lyapunov 19
statically determinate 11
statically indeterminate 11, 58
stiffness 6
stiffness tensor (see elastic
tensor)
strain 6, 31-32
strain tensor 92

strain
  axial 7, 55
  axysymmetric 42
  compatibility 39, 110
  elastic 94
  equi-biaxial 102
  finite 94
  gauge 97
  gauge rosette 97
  incompressible 108
  incompressible 33, 95-96
  large 94
  normal 33, 97, 102, 108
  one-dimensional 6
  plane 41, 115
  principal 33, 95, 105, 107, 115
  proof (see yield)
  pure shear 45, 115
  quasi-static 32
  shear 33, 98, 102, 105, 108
  small 33, 94
  state 102
  symmetry 94
tensor 35-36, 38, 41, 43, 92-95, 104, 107-110,
113
tensor 1st invariant 115
three-dimensional 40
trace 115
transverse 7
two-dimensional 43
uniaxial 109-110
volumetric 33
yield 58
stress 3, 24
stress state 39
stress tensor 39
stress
average 88
axysymmetric 42
biaxial 84, 89, 115
compressive 10, 27, 84
continuum 13
discontinuity 53
distribution 83
equilibrium 38, 42, 81-83, 110, 112
equilibrium equations 28
field 80-84
hydrostatic 101
matrix 26, 84, 114
maximum shear 120
non-uniform 82-84
normal 23, 27, 89, 98, 102-103, 108
one-dimensional 3
plane 39, 115
pressure 108, 114
principal 91, 98, 103-104, 107, 113, 115, 117
profile 18
pure shear 45, 115
shear 23, 27, 83, 89, 98, 102-103, 108
sign convention 82
state 23, 26, 102
symmetry 26, 29, 105
tensile 10, 27, 84
tensor 1st invariant 115
trace 108, 115
traction 38, 81-84
triaxial 109
two-dimensional 43
uniaxial 9, 58, 81-83, 88, 106-107, 109-110
uniform 81
vector 4, 24, 89, 98, 112
yield 18
stretch 32

T
tensor
component 106
components 36, 105
derivative 110
equality 30
function 35-36

U
unit tensor 76
unit tensor (see also rank 2 tensor identity)

V
vector 120
acceleration 3
cross product 28, 85
derivative 110
flux 114
force 3, 80
position 3
stress 4
velocity 3
velocity 3
volume 85, 96
change 96
infinitecimal 80

Y
Young’s modulus 38, 55, 107
Young’s modulus (see elasticity)
1. Abbreviations

BC - boundary conditions
BTW - by the way
CS - coordinate system
CT - coordinate transformation
iff - if and only if, also ↔
R2T - rank 2 tensor
wrt - with regards to, with respect to

2. Notation

In this course a mix of tensor and index notations is used. Although some violation of this notation convention is unavoidable, mostly italic is used for scalar variables, e.g. \( \lambda \). Vectors and tensors of rank 2 are shown either in bold or with indices, e.g. the coordinate vector \( \vec{x} \) can appear as \( \mathbf{x} \) or as \( x_i, i = 1, 2, 3 \).
Typically capital letters are used for R2T, e.g. \( \mathbf{T} = T_{pq} \). Tensors of rank 4 are typeset in Helvetica, either in bold, when used with other tensors and vectors, or with indices. For example the elasticity (stiffness) tensor can appear as \( \mathbf{C} \) or as \( C_{ijkl} \). Italic is also used to emphasise key terms or concepts.

\( ↔ \) means if and only if, also iff.

\( \cdot \) means inner product or dot product, contraction on the inner index, e.g. the square of the length of vector \( \vec{x} \) is \( \mathbf{x} \cdot \mathbf{x} \).

\( : \) means tensor product, contraction on the 2 inner indices, e.g. \( \mathbf{A} : \mathbf{A} = A_{ij}A_{ij} \).

\( \otimes \) is a dyadic product, e.g. \( \mathbf{x} \otimes \mathbf{x} = x_i x_j = A_{ij} \).

3. Recommended reading

All books are available from either the Queen’s building library, the Physics library or the Chemistry library. Some books have multiple copies.

The following books are recommended for understanding the idea of a tensor, index notation, the ideas of stress, strain and deformation, ideas of principal strain and stress, elasticity, etc.

Easy/medium introductory level:
→ (Fleisch, 2012) a gentle intro to vectors and tensors with lots of practical examples and illustrations.
→ (Gere, 1997) very easy, chapter 7, inferior notation, stress/strain analysis, little or no tensors. Many copies.
→ (Mase, 1970) easy/medium, chapters 1-3, lots of solved problems and further problems for self-study.
→ (Tadmor, 2012) the beginning is a very gentle introduction to stress and strain, the need for tensors and index notation.
→ (Fitzgerald, 1982) easy, sections 6.5 to 6.15, no tensors, just stress/strain analysis.
→ (Case, 1999) very easy, beginning of chapter 5, beware - inferior notation, stress/strain analysis, no tensors.
→ (Beer, 2009) very easy, sections 6.1 to 6.6, no tensors, stress/strain analysis.
→ (Crandall, 1978) easy and quite thorough, stress/strain, little or no tensors.
→ (Byars, 1963) very easy, chapters 1-2, no tensors
→ (Hosford, 2005) and (Hosford, 2010) easy, chapter 1 - stress/strain, some tensor language. Multiple copies.
→ (Valliappan, 1981) explains 3D stress and strain in chapters 1 to 3, although he cuts corners and simplifies things a bit. He never mentions tensors, all is done with matrices.
(Calcote, 1968) is ok. Sections 2.1, 2.5-2.7 explain tensors and index notation. Section 3 explains stress, strain and elasticity. Can skip more advanced material.

(Spencer, 1980) chapters 1-5, index notation, tensors, stress, strain. Skip more advanced material. Beware: some notation is different to mine. Another copy is in physics library.

(Bonet, 2008) is an advanced read overall, but some sections are explained quite easily. Try sections 2.2, 4.4, 4.5, 5.2 - tensors and stress.

Intermediate level:

(Reedy, 2013) a very good book overall.

(Rees, 2000) easy, but misleading due to inferior notation, chapter 13, very applied, use with caution.

(Ford, 1963) easy/medium, parts 1-2, skip curvilinear coordinates, lots of copies.

(Bourne, 1992) chapters 1, 2, 8, skip vector and tensor calculus. Only about tensors and notation, a little bit about stress, no strain. Multiple copies.

(Malvern, 1969) a classic book, you only the beginning - stress/strain analysis and tensor basics.

(Brillouin, 1964) chapters 1, 2 - slightly outdated notation, skip covariant/contravariant and curvilinear sections.

(Scipio, 1967) part 1 contains all you need: index notation, tensors, stress, strain. Beware: some notation is different to mine. Another copy is in physics library.

(Sokolnikoff, 1956) the beginning is suitable as an introduction to stress, strain, tensors and index notation.

Advanced level:

(SEgel, 1987) chapters 1-2 - index notation, tensors. Skip the theorems and more advanced material.

(Chorlton, 1976) chapters 12-13, tensors/stress

(Hunter, 1976) chapter 4 - notation, chapter 5 - strain.

(Chaves, 2013) has lots of nice diagrams. Chapter 1 - tensors, chapter 3 - stress. Skip chapter 2 on strain - too advanced.

(Fung, 1969) is a demanding but still a very good book. Read chapters 1-5, skip more advanced material.

(Chadwick, 1976) and (Hodge, 1970) are more advanced, with different notation, but still useful, if one wants to know more.

(Eringen, 1967) a classic book on the subject

(Akivis, 2003) extra knowledge of tensors and tensor calculus.

(Sokolnikoff, 1951) another classic book, the beginning is not too hard, and very relevant.
4. One-dimensional stress/strain analysis

4.1. Fundamental ideas of stress and strain and linear elasticity

4.1.1. Stress

We consider as known and understood the concepts of force, \( \mathbf{f} \), position vector, \( \mathbf{x} \), velocity, \( \mathbf{v} = \frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} \), acceleration \( \mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{x}} \) and coordinate system.

We also consider as known and understood the Newton’s laws, e.g. the second law of Newton can be written as:

\[
\mathbf{f} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}
\]  
(1)

or, if \( m \) is not constant, then:

\[
\mathbf{f} = \frac{d(m\mathbf{v})}{dt}
\]  
(2)

or

\[
d(m\mathbf{v}) = \mathbf{f}dt
\]  
(3)

meaning that the total force acting on a body causes the body to accelerate, with the acceleration vector being proportional to the force vector, Eqn. (1). Or we can say that the force is proportional to the rate of change of the linear momentum, Eqn. (2). Or we can say that the linear momentum is always conserved, Eqn. (3), i.e. the change in the linear momentum can be caused only by the impulse, the product of force by time.

If \( d\mathbf{v} = 0 \), then the total force is \( \mathbf{f} = 0 \). We call this force equilibrium.

We introduce a concept of a moment. If force \( \mathbf{f} \) acts in point A, then the moment of \( \mathbf{f} \) about some other point B is the product of the force by the distance from B to the line of action of \( \mathbf{f} \). This distance, \( h \), is typically called a shoulder. The moment, \( \mathbf{m} \), is a vector.

\[
\mathbf{m} = \mathbf{f}h
\]  
(4)

(Ex. prob. 1).

If \( \mathbf{m} = 0 \) then we have a moment equilibrium.

Consider a pulley lifting mass \( m \).

I want to know the minimum diameter of the rope, required for lifting the mass.

First we need to introduce the idea of a free body diagram, which is contructed using the sectioning method. In this method we introduce imaginary cuts to the system to isolate (free) body or bodies of interest. We substitute the actions of removed bodies by forces and moments. I imaginary cut the rope at two
points and substitute the action of missing parts by forces and moments.

\[ p = u = r = v = mg. \]

I also introduce a coordinate axis, 1 or \( x \), with a positive direction defined.

Finally I need the third law of Newton, i.e. "action=-reaction"

Since all parts are in static equilibrium, I quickly conclude that: \( q = t = s = w = 0 \), \( p = u = r = v = mg \). So the force in the rope is \( mg \) along its axis. Since my cuts were in arbitrary positions along the rope, I conclude that the force in the rope is everywhere \( mg \).

Graphically the force profile this can be shown as this.

We can now introduce the concept of a stress vector, \( t \). If the cross-section of the rope is \( A \), and the axial force is \( f \), then I define 1D stress as:

\[ t = \frac{f}{A} \]  

(5)

In the one-dimensional class of problems, like the rope, where there is only a single spatial dimension, the stress and force are always along the axis. Hence we can often neglect the vectorial nature of the stress vector, and simply use the term stress.

\[ t = \frac{f}{A} \]  

(6)

By convention, tensile stress is positive and compressive stress is negative.

\( t \) is really a pressure. The units of stress are those of pressure - force per unit area. In SI the units of stress are Pascal, abbreviated as Pa. 1 Pa = 1 N / 1 m\(^2\). 1 Pa is a very low stress in the majority of engineering applications. Typically we use MegaPascal, MPa, which can be conveniently defined as

\[ 1 \text{ MPa} = 1 \text{ N} / 1 \text{ mm}^2 \]

Some typical values of stress or pressure are: bicycle tyre - 0.5 MPa, car tyre - 0.2-0.3 MPa, tensile strength of nylon - 70 MPa and of polyamide - 110 MPa, steel yield stress - 300-700 MPa. (The concepts of the strength and yield stress will be discussed in detail in Properties of Materials. These are fundamental properties of engineering materials).

Why are bike tyre pressures higher than in car tyres, even though cars are much heavier than bikes?

Because car tyres are much wider than in bikes. So car tyres have a much bigger contact area with the road, which allows to reach the required reaction force.

Back the rope problem. The magnitude of axial stress in the rope is \( t = \frac{mg}{A} \).

What’s so clever about the idea of stress? Stress is a relative measure of force. It is force related to the area over which it acts. If I imaginary split the rope into several ropes of smaller cross sectional area,
e.g. 4, then the force in each rope will be \( f/4 \), whereas stress remains the same, \( t \), because the cross section area in each "sub-rope" has decreased proportionally.

We can now pose and solve simple solid mechanics (or mechanics of materials) problems on the pulley. Let’s assume the rope is of a circular cross section.

**Problem A:** What is the minimum diameter of rope, used in a simple pulley, for the stress in the rope not to exceed 100 MPa while lifting a mass of 200 kg?

The stress in the rope is \( t = mg/A \), where \( A = \pi d^2/4 \), \( d \) is the diameter. So

\[
\frac{mg}{t} = \frac{\pi d^2}{4} \rightarrow d = \sqrt{\frac{4mg}{\pi t}}
\]

The smallest \( d \) will result from the highest possible \( t \), which is 100 MPa. If we use MPa as the unit of stress, kg as the unit of mass and m/\( s^2 \) for the freefall acceleration, then the units for diameter will be mm. Assuming \( g = 10 \, m/\, s^2 \):

\[
d = \sqrt{\frac{4 \times 200 \times 10}{3.14 \times 100}} = 5
\]

So the answer is 5 mm.

**Problem B:** If the stress in the rope must not exceed 200 MPa, what is the maximum load that a pulley with a rope of diameter 3 mm can lift?

This is the opposite problem to problem A. All we need to do is rearrange Eqn. (7).

\[
m = \frac{\pi d^2 t}{4g} = \frac{3.14 \times 3^2 \times 200}{4 \times 10} = 141.3
\]

So the answer is 141.3 kg.

**Problem C:** What will be the maximum stress in a rope of 10 mm diameter lifting a mass of 2 tons?

Again, I just rearrange Eqn. (7).

\[
t = \frac{4mg}{\pi d^2} = \frac{4 \times 2 \times 10^3 \times 10}{3.14 \times 10^2} = 255
\]

The answer is 255 MPa.

The follow-up to problem C is **Problem D:** Choose an engineering material for the rope, that can carry the stress of 255 MPa.

This problem is outside the scope of this course. It is addressed fully in the Properties of Materials course.
4.1.2. Strain

A pulley rope will elongate under the application of load. The higher the load - the greater the elongation. In fact, under certain conditions, the elongation is proportional to the force. This is the famous law of elasticity, first discovered in this form by Robert Hooke in 1660, when he published it as an anagram: "ceiinosssttu". In 1678 he published the solution to the anagram: "ut tensio, sic vis", which can be translated as "as the extension, so the force".

If the original length of the rope on one side of the pulley was \( l_1 \), then after mass \( m \) was attached to the pulley, the length of the rope increased to \( l_2 \). The elongation is \( \Delta l = l_2 - l_1 \).

Since the stress in the rope is \( mg \), we can say that
\[
\Delta l k_1 = mg
\]
where \( k_1 \) is the stiffness of the rope of length \( l_1 \).

Although the above equation has some use for calculating elongations, it has a major problem - \( k \) depends on the length of the rope. Indeed, if the rope is initially twice as long, under the application of the same load it’s elongation will be twice as high. It would be good to have stiffness as a material property, that does not depend on the geometry of the problem.

To address this problem, we introduce the idea of strain, which is a relative measure of elongation, similar to the idea of stress as a relative measure of force. By definition, a one-dimensional strain is
\[
e = \frac{\Delta l}{l}
\]
where \( l \) is the original undeformed length.

Strain is dimensionless.

By convention, tensile strain is positive and compressive strain is negative.

If we fix a coordinate axis with the rope, \( x \), then we can talk of displacement, \( u \), which is by how much each point of the rope moved from the origin. We can choose the origin arbitrarily, but let’s take it, for simplicity, at the point where the rope leaves the pulley roller. The displacement profile will look like this.

\[
\Delta l
\]

It is of key importance that the displacement profile is linear.

We can now redefine \( e \) from Eqn. (9) as
\[
e = \frac{u}{x}
\]
Since \( u \) is a linear function of \( x \), \( e \) is constant along the length of the rope. We can say that the
displacement *gradient* is constant. Strain \( e \) is really a displacement gradient, or relative displacement.

### 4.1.3. Young’s modulus

For **linearly elastic materials**

\[
t = E e
\]

(11)

where \( E \) is the *Young’s modulus*, named after Thomas Young (1773-1829), an English scientist.

Since \( e \) is dimensionless, \( E \) has the units of pressure. For engineering materials we usually use MPa or GigaPascal, GPa. Some typical values are: steel - 200 GPa, aluminum alloys - 70 GPa, brass - 100 GPa, concrete - 30 GPa, nylon - 2 GPa, wood - 9 GPa.

With the ideas of strain and elasticity added to the idea of stress, some further pulley problems can be posed and solved.

**Problem E:** What should be the minimum diameter of a steel wire, used for the pulley rope, of initial length of 20 m, so that it does not extend more than 10 mm under the application of mass of 500 kg?

Let’s assume the Young’s modulus of 200 GPa for steel. From Eqn. (9) strain is

\[
e = \frac{\Delta l}{x} = \frac{10}{20 \times 10^3} = 5 \times 10^{-4}
\]

From Eqn. (11) stress, in MPa, is

\[
t = E e = 2 \times 10^5 \times 5 \times 10^{-4} = 100
\]

Substituting \( t \) into Eqn. (7) we can calculate the diameter:

\[
d = \sqrt{\frac{4mg}{\pi t}} = \sqrt{\frac{4 \times 500 \times 10}{3.14 \times 100}} \approx 8
\]

Since we use value for stress in MPa, the answer is in mm - 8 mm.

**Problem F:**

Calculate the maximum extension of a nylon pulley rope of initial length of 3 m and 1 mm diameter, under the application of mass of 2 kg.

First we need to calculate the stress in the rope. Using Eqn. (8):

\[
t = \frac{4mg}{\pi d^2} = \frac{4 \times 2 \times 10}{3.14 \times 1^2} = 25
\]

Since \( d \) was in mm, the stress in in MPa - \( t = 25 \) MPa. Now, let’s calculate strain in the rope using Eqn. (11). Let’s assume \( E = 2 \) GPa for nylon.

\[
e = \frac{t}{E} = \frac{25}{2 \times 10^3} = 1.25 \times 10^{-2}
\]

Now the elongation can be calculated from Eqn. (9). Using mm:

\[
\Delta l = le = 3 \times 10^3 \times 1.25 \times 10^{-2} = 37.5
\]

### 4.1.4. Poisson’s ratio

A closer look at the pulley example reveals the first complication with our one-dimensional strain theory developed so far. Ropes made of most ordinary materials will contract in *transverse* direction when extended in the axial direction.
So, in addition to the axial strain, we now need to have a definition for the transverse strain. Similar to the axial strain we take it as a ratio of the change in diameter, \( \Delta d = d_1 - d_0 \), to the original diameter.

Let’s add another two coordinate axes - 2 in the plane of the paper, and 3 normal to the paper, as shown above.

We now use subscript 11 for axial strain, \( e_{11} \), and 22 and 33 for two transverse strains, \( e_{22}, e_{33} \).

\[
e_{22} = e_{33} = \frac{\Delta d}{d_0} \tag{12}
\]

If we take the origin in transverse direction at the axis of the rope, then \( u_2 \) is the transverse displacement of any point in the rope along axis 2, and \( u_3 \) is the transverse displacement along axis 3. So the definitions of the transverse strains can be rewritten as:

\[
e_{22} = \frac{u_2}{x_2} \tag{13}
\]
\[
e_{33} = \frac{u_3}{x_3} \tag{14}
\]

The profiles of the transverse displacement and strain, e.g. along 2, are simple:

\[
\frac{\Delta d}{2} \quad \frac{\Delta d}{2}
\]

Hence, transverse strain is constant along the diameter.

It turns out the degree to which ropes contract in transverse direction is described by a single material parameter, \( \nu \), the Poisson’s ratio, named after Siméon Denis Poisson (1781-1840), a French mathematician. By definition:

\[
\nu = -\frac{e_{22}}{e_{11}} = -\frac{e_{33}}{e_{11}} \tag{15}
\]

The typical values of the Poisson’s ratio for common engineering materials are: steel - 0.33, Aluminium alloys - 0.33, concrete - 0.1, rubber - 0.49.

The Poisson’s ratio of cork is 0. Why does this make it an ideal material to close bottles?
4.1.5. Displacement

A careful look at Eqns. (10), (13) and (14), will show that all three strain measures defined so far, axial and transversal, have the same structure. We mentioned before that strain is a gradient of displacement, which we now write explicitly using differentiation.

\[ e_{11} = \frac{du_1}{dx_1}; \quad e_{22} = \frac{du_2}{dx_2}; \quad e_{33} = \frac{du_3}{dx_3} \]  

(16)

From Eqn. (16) displacements can be calculated via integration:

\[ u_1 = \int_{x_1} e_{11} dx_1; \quad u_2 = \int_{x_2} e_{22} dx_2; \quad u_3 = \int_{x_3} e_{33} dx_3 \]  

(17)

So a linear displacement profile means a constant corresponding strain and vice versa.

4.1.6. Elastic energy

When bodies deform internal forces do work on corresponding displacements. Or we can say that stresses do work on corresponding strains. If \( x_1 \) is along the axis of the rope in tension (or a column under compression), then the stress is \( t \) and the corresponding strain is \( e_{11} \). The work is an integral of stress over strain, which for linear elasticity, \( t = E e_{11} \), is just the area of a triangle.

\[ H = \frac{1}{2} t e_{11} \]  

(18)

Hence the stored elastic energy is

Note that the units of \( H \) are the units of stress. In other words \( H \) is the elastic energy per unit of volume, J / m\(^3\) or N / m\(^2\). To calculate the elastic energy for the whole body, one has to integrate \( H \) over volume. If \( H \) is constant everywhere in the body, then the total elastic energy is simply \( HV \), where \( V \) is the volume of the body.

Elastic material is just like a conventional spring - when loads are applied, the body is deforming and the external work is transferred into stored internal elastic energy. When the body is allowed to relax back to its original configuration, the stored elastic energy is transferred back into work. A watch spring a simple analogy to an elastic material.

(Ex. probs. 2, 3, 4, 5.)

4.2. Pin-joined frame

Pin-joined frame is an idealised engineering structure in which there are no moments in the joints, and hence no bending moments anywhere. Some very complex stress/strain fields will exist in the immediate vicinity of the pins. If these are ignored, then the stress state in each bar is one-dimensional, or uniaxial, tension or compression. These structures are easy to analyse.

Consider a pin-joined frame structure in 2D space, loaded by force F:
From moment equilibrium it follows that forces in the bars are always along bar axes. The free body diagrams for this problem look like this:

where both reaction forces are found from equilibrium equations. It’s clear that in this problem $R^2$ is tensile (positive) and $R^1$ is compressive (negative).

Let’s use superscripts 1 or 2 to denote the bar number. The stresses in bars are $t^1 = R^1/A^1$ and $t^2 = R^2/A^2$, where $A^1$ and $A^2$ are the cross sections of the two bars.

If in each bar $x_1$ is along the axis, and if both bars are made of the same material, then

$$e^1_{11} = \frac{R^1}{EA^1}$$

$$e^2_{11} = \frac{R^2}{EA^2}$$

Axial displacements are found by integration.

$$u^1_x = \frac{1}{EA^1} \int R^1 dx_1 + C$$

$$u^2_x = \frac{1}{EA^2} \int R^2 dx_1 + D$$

where $C$ and $D$ are integration constants found from the boundary conditions - displacements are zero at the constraints.

For bar 1: $u^1(x_1 = 0) = 0 \rightarrow C = 0$.

For bar 2: $u^2(x_1 = 0) = 0 \rightarrow D = 0$.

Finally

$$u^1 = \frac{R^1}{EA^1} x_1$$
\[ u_2^2 = \frac{R^2}{EA^2} x_1 \]

The maximum displacements are clearly at the point where the force is applied:
\[ u_{1\text{max}}^1 = \frac{R^1 L^1}{EA^1} \]
\[ u_{1\text{max}}^2 = \frac{R^2 L^2}{EA^2} \]

However, if the displacement is substantial, then one also has to consider the rigid body motion:

In this problem this is only rigid body rotation. There is no rigid body translation.  
(Ex. prob. 6.)

Note that solution of problems of this sort relies on the ability to solve for all reaction forces and/or reaction moments only from the equations of force and/or moment equilibrium. Such problems are called *statically determinate*. Problems where it is not possible to calculate reaction forces and/or moments only from the equations of equilibrium are called *statically indeterminate*.

### 4.3. Statically indeterminate systems

Consider a force applied to the end point of \( n \) wires:

There are only two useful equation of equilibrium we can use from statics of rigid bodies - sum of the forces in 2 directions (in the plane of drawing), e.g. vertical and horizontal, is equal to zero. However, there are \( n \) unknown reaction forces. We need a further \( n - 2 \) equations.  
These can only come from some other information found in the problem. In this particular example we'd exploit the fact that displacements of the end points of all wires are equal: \( u^1 = u^2, u^1 = u^3, \ldots, u^1 = u^n, n - 1 \) equations altogether. So the problem becomes solvable.

(Ex. prob. 7.)
4.4. Euler-Bernoulli (slender beam) bending theory

Mathematically a beam is a cylinder of arbitrary cross section \( A \), i.e. if \( A \) is moved along a straight line, normal to \( A \), then the bounding contour of \( A \) will describe a surface of a beam.

Without the loss of generality let’s assume the beam is bent about axis 3:

A beam is called *slender* if some characteristic dimension of the cross section is much smaller than the length, e.g. if \( \sqrt{A} \ll L \), where \( L \) is the length of the beam.

The specific theory we study in this section, the Euler-Bernoulli theory, has several further assumptions, which make the solution of a bending problem particularly simple:

- There is a neutral line in the beam, which passes through the same point in any cross section normal to the beam axis. The normal line does not change length.
- The deflections are small. This is required to ensure that the previous assumption is valid.
- Planar sections which are initially normal to the axis of the beam remain planar and normal to the axis throughout bending;
- Any straight line normal to the neutral axis remains straight throughout the deformation.
- The body force is negligible.
- The deformation is slow.
- Only tensile or compressive deformation is allowed.

Together all above assumption mean that the stress state at any point is described only by the axial stress \( t \), and strain is described by the axial and the two transverse strains, \( e_{11}, e_{22}, e_{33} \). If the bar is imaginary split into multiple thin *fibres*, running along its axis, then the behaviour of each such fibre can be described by one-dimensional theory of Sec. 4.1.

Usually, in a bending problem, one is interested in finding *deflections* resulting from the application of external loading. We designate deflection of the neutral line as \( w \).

Let’s consider an infinitesimal length of the beam, \( dx_1 \), separated by two cross sections, in deformed configuration.
Our assumptions lead to the following expressions for displacements:

\[ u_1 = x_2 \tan \alpha = x_2 \frac{dw}{dx_1} \quad (19) \]

We use full differentials because \( w \) is a function of only \( x_1: w = w(x_1) \).

\[ u_2 = w + \int e_{22} dx_2 \quad (20) \]

\[ u_3 = \int e_{33} dx_3 \quad (21) \]

Let’s consider force and momentum equilibrium for a "slice" of thickness \( dx_1 \):

where \( q \) is distributed load, i.e. load per length.

Note that in all problems studied so far, axial stress \( t \) was constant along the cross section. However, in bending problems \( t \) will change along \( x_2 \). This means our previous definition, \( t = f/A, \) Eqn. (5), is no longer adequate. We need a new, continuum mechanics, definition of stress, i.e. for stress as a continuously changing function.

\[ t = \lim_{\Delta A \to 0} \frac{f}{\Delta A} \quad (22) \]

Exactly how it changes will come out as part of this analysis. Cross section force and moment change smoothly along \( x_1 \), hence the use of \( dF \) and \( dM \). Sum of forces along 2 is zero:

\[ dF = qdx_1 \quad \rightarrow \quad q = \frac{dF}{dx_1} \]

Sum of moments about C is zero:

\[ dM = Fdx_1 + qdx_1 \frac{dx_1}{2} \]

or, neglecting the quadratic term,: 

\[ dM = Fdx_1 \quad \rightarrow \quad F = \frac{dM}{dx_1} \quad \rightarrow \quad q = \frac{d^2 M}{dx_1^2} \quad (23) \]
where

\[ M = \int tdAx_2 \]  \hspace{1cm} (24)

dA is the infinitesimal cross section area, \( tdA \) is the infinitesimal force normal to the cross section, and \( tdAx_2 \) is the moment of this infinitesimal force about the neutral axis. The integral sums all such infinitesimal moments to give the total moment in the cross section.

From Eqn. (19):

\[ e_{11} = \frac{du_1}{dx_1} = x_2 \frac{d^2w}{dx_1^2} \]

From one-dimensional Hooke’s law:

\[ t = E e_{11} = E x_2 \frac{d^2w}{dx_1^2} \]  \hspace{1cm} (25)

This means that \( t \) is a linear function of \( x_2 \). This is a very important result of this bending theory.

So that

\[ t = t_{max} \frac{x_2}{x_{2}^{max}} \]  \hspace{1cm} (26)

Putting this into (24):

\[ M = t_{max} \frac{x_2}{x_{2}^{max}} \int x_2^2dA \]

Note that the integral depends solely on the cross section. It is called the second moment of area and is denoted \( I \):

\[ I = \int x_2^2dA \]  \hspace{1cm} (27)

So that

\[ t_{max} = \frac{Mx_{2}^{max}}{I} \]  \hspace{1cm} (28)

By combining (28), (26) and (25):

\[ \frac{d^2w}{dx_1^2} = \frac{M}{IE} \]  \hspace{1cm} (29)

This is the famous ordinary differential equation (ODE) of 2nd order, linking the curvature with the bending moment, the mechanical properties of material (the Young’s modulus) and the property of the cross section (the second moment of area).

The second derivative of deflection is curvature, the inverse of the radius of curvature, \( \rho \):

\[ \frac{d^2w}{dx_1^2} = \frac{1}{\rho} \]

Combining (29) with (23):

\[ q = \frac{d^2}{dx_1^2} \left( IE \frac{d^2w}{dx_1^2} \right) \]  \hspace{1cm} (30)
This is the famous 4th order ODE, suitable for variable $I$ and $E$. If these are constant along the axis of the beam, then:

$$q = IE \frac{d^4w}{dx_1^4} \tag{31}$$

It is interesting to note that the first 4 derivatives of deflections all have easy mechanical meaning:

$$\frac{dw}{dx_1} = \tan \alpha = \alpha \quad \text{angle, slope, gradient}$$

$$\frac{d^2w}{dx_1^2} = \frac{1}{\rho} \propto M \quad \text{curvature, bending moment, axial stress}$$

$$\frac{d^3w}{dx_1^3} \propto F \quad \text{shear force}$$

$$\frac{d^4w}{dx_1^4} \propto q \quad \text{distributed load}$$

What this means is, if one is able to capture the deflection of the beam, then the stress/strain state can be calculated via numerical differentiation, e.g. graphically.

For uniaxial stress state

$$e_{22} = e_{33} = -\nu e_{11}$$

Since $e_{11} = E\iota$, and $\iota$ is positive on one side of the neutral line and negative on the other, therefore $e_{22}$ and $e_{33}$ are of different signs either side of the neutral line. For example, an initially square cross section might deform after bending like this:

Above the neutral line the material is in tension, hence $e_{22} = e_{33} < 0$, and the width and the height of the cross section will decrease. Below the neutral line the material is in compression, hence $e_{22} = e_{33} > 0$, and the width and the height of the cross section will increase.

The Euler-Bernoulli theory becomes exact if there is no shear force acting on the cross section. Otherwise, the theory is approximate, and the degree of deviation of this theory from experiment is directly related to the magnitude of the shear force. This special case of zero shear force is called pure bending, and is usually achieved experimentally with a four-point bending setup like this:
The added advantage of this geometry is a constant bending moment between the innermost rollers. This fact is widely exploited in laboratory materials testing experiments.

Eqn. (28) shows that the maximum stress in a cross section depends on $x_{\text{max}}^2$ and $I$ - two of the properties of the cross section. This means we need to examine the properties of cross sections in more detail. This is the subject of the next section.

(Ex. prob. 8).

4.4.1. Properties of cross sections

Consider a cross section as an arbitrary 2D area. It can be regular or irregular, with a single or multiple boundaries. We use a coordinate system (CS), with arbitrary orientation and origin O. We denote this CS as $x_j$, where $j$ can be 1 or 2. So the coordinate axes are $x_1$ and $x_2$.

By definition the first moments of area in $x_j$ are:

$$i_1 = \int x_2 dA$$  (32)

$$i_2 = \int x_1 dA$$  (33)

where $dA$ is an infinitesimal element of area located at position $x_1$, $x_2$, and the integration is done over the whole area. So the moments depend on CS!

In another CS shifted by $S_j$:

$$x'_{j} = x_j - S_j$$  (34)

the first moments become:

$$i_1' = \int x_2' dA = \int (x_2 - S_2) dA = i_1 - S_2 A$$  (35)

$$i_2' = \int x_1' dA = \int (x_1 - S_1) dA = i_2 - S_1 A$$  (36)
By definition centroid, \( C \), is a point, such that if the origin of a CS is put there, then both first moments of inertia vanish.

From Eqns. (35) and (36), setting \( i_1' = i_2' = 0 \), one finds \( S_j \), the coordinates of \( C \) in \( x_j \):

\[
S_1 = \frac{i_2}{A} \quad (37)
\]

\[
S_2 = \frac{i_1}{A} \quad (38)
\]

If cross section had thickness and constant density, then the centroid would coincide with the centre of mass.

By definition, the second moments of area are

\[
I_{11} = \int x^2_1 dA 
\]

\[
I_{22} = \int x^2_1 dA 
\]

\[
I_{12} = \int x_1 x_2 dA 
\]

\[
I_r = \int r^2 dA 
\]

where \( r^2 = x^2_1 + x^2_1 \). \( I_r \) is called the polar second moment of area. Often \( J \) is used instead of \( I_r \).

(Ex. prob. 9.)

Although the moments can be calculated wrt any CS, for bending one needs the second moments wrt the neutral axis. This means that one needs to be able to calculate the second moments wrt \( C \).

(Ex. probs. 10, 11, 12, 13, 14, 15, 16, 17.)

Note that the units of second moments of area are \([I] = L^4\).

The moments of complex cross sections can be calculated by splitting the original section into easy to integrate areas. A hole, or a cutout, is a negative area. In general, if \( A = \sum_{j=1}^N A_j \), then

\[
i_1 = \sum_{j=1}^N i_1^j \quad (43)
\]

where

\[
i_1^j = \int_{A_j} x_2 dA; \quad i_2^j = \int_{A_j} x_2 dA; \quad \cdots \quad i_N^j = \int_{A_j} x_2 dA \quad (44)
\]

And the centroids of each sub-area are

\[
S_1^j = \frac{i_1^j}{A_j}; \quad S_2^j = \frac{i_2^j}{A_j}; \quad \cdots \quad S_N^j = \frac{i_N^j}{A_j} \quad (45)
\]

From (45) and (43)

\[
S_2 = \frac{i_1}{A} = \sum_{j=1}^N \frac{S_1^j A_j}{A} = \frac{1}{A} \sum_{j=1}^N S_2^j A_j \quad (46)
\]

\( S_1 \) is calculated similarly.

The same applies to the second moments, e.g. for \( I_r \)

\[
I_r = \sum_{j=1}^N I_r^j \quad (47)
\]

where

\[
I_1^j = \int_{A_j} r^2 dA; \quad I_2^j = \int_{A_j} r^2 dA; \quad \cdots \quad I_N^j = \int_{A_j} r^2 dA \quad (48)
\]
Care should be taken to refer the second moments of each sub-section to the centroid of the whole section.

(Ex. prob. 18).

### 4.4.2. Plasticity in bending

Eqn. (28) says that $t_{\text{max}} \propto M$, so with increasing $M$ stress eventually reaches plastic limit. The constitutive model at that point switches from elasticity to some plastic behaviour. The exact description of plasticity is beyond the scope of this course. Popular criteria for the onset of plasticity, such as due to von Mises or Tresca have very simple formulation for uniaxial stress state - when $t = \sigma_Y$, where $\sigma_Y$ is the yield stress, then plasticity starts.

The stress profile is linear, with zero stress always on the neutral line. With increasing moment, stress increases maintaining a linear profile, until the maximum stress reaches yield value, $\sigma_Y$. The diagram shows 5 stress profiles, with $M$ increasing from profile 1 to profile 5.

![Stress Profile Diagram](image)

*Plastic flow* first starts where $x_2 = x_2^{\text{max}}$, i.e. the layer of material furthest from the neutral line. The 2nd stress profile is drawn at that moment.

As $M$ keeps increasing, plasticity progresses inwards towards the neutral line. Eventually $|t|$ on the other side will reach $\sigma_Y$ and plastic flow will start there too. The 4th stress profile is drawn at that moment.

Note that this diagram is nothing but qualitative. Many things might happen during plastic flow, including a shift of the neutral line.

Conclusion: if one wants to avoid plasticity in bending, one must ensure that

$$\frac{M x_2^{\text{max}}}{I} < \sigma_Y$$

(Ex. prob. 19).

### 4.5. Stability

Consider a pendulum under the action of a gravitational force. The pendulum has two equilibrium positions - one with the pendulum directly below the support (left) and the other with the pendulum directly above the support (right):
However, only the left configuration is \textit{stable}. The right configuration is \textit{unstable}. This can be verified easily by considering a small angular perturbation, or disturbance, \( \alpha \), from the ideal vertical alignment of the pendulum. In the stable configuration (left) there will be a horizontal force, \( mg \sin \alpha \cos \alpha \), returning the pendulum to the equilibrium position. In the unstable configuration (right) horizontal force of the same magnitude pushes the pendulum away from the equilibrium position.

The idea of linking the concept of stability to a perturbation can be mathematically formalised as follows.

Consider function \( f(x) \) which describes some system. This function has a specific boundary, or initial, condition at \( x = 0 \): \( f(0) \). If this initial condition is perturbed slightly, so that the value of the function is \( \hat{f}(0) \), then the evolution of the system with \( x \) might be different. The following stability definition is due to Lyapunov. \( f(x) \) is stable, if

\[
\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \exists \ \text{such that,} \ |f(0) - \hat{f}(0)| < \delta, \text{ then } |f(x) - \hat{f}(x)| < \varepsilon \forall x \geq 0
\]  

(50)

where that maths symbols are \( \forall \) - for all, \( \exists \) - there exist(s), \( \delta \) - such that.

Consider adding a horizontal supporting elastic spring of stiffness \( k \) to help stabilise the inverted pendulum:

Consider that the pendulum length is \( l \). For a small horizontal perturbation \( x \), there is now a horizontal restoring force, pushing the pendulum back to the equilibrium position. The moment of the weight is \( mgx \). The restoring moment from the spring is \( kxl \). By comparing these moments, one can see that if \( kl > mg \),...
then the system will be restored to the vertical equilibrium position. We can say that under this condition the vertical equilibrium is stable. If \( kl < mg \), then the spring will not be able to restore the pendulum to vertical equilibrium. We can say that under this condition the vertical equilibrium becomes unstable. If \( kl = mg \), then a new equilibrium position is possible, with the pendulum deviating from the vertical position. This equilibrium is stable.

Condition \( kl = mg \), which separates stable and unstable regions for the initial equilibrium configuration, is called the critical condition. In most practical applications one wants to find a critical condition for a particular equilibrium configuration.

For the inverted pendulum example the equilibrium regions can be summarised as follows.

- \( kl > mg \); stable
- \( kl < mg \); unstable
- \( kl = mg \); critical, new equilibrium possible

Many structures can exhibit an elastic loss of stability - shells, beams, etc. Elastic instability can be catastrophic, but also useful. Simple devices such as hair clips or light switches rely on an elastic loss of stability, or, more accurately, on the existence of multiple distinct stable elastic configurations separated by unstable regions. The simplest practical case is stability of columns under axial compression. Instability in such cases is called buckling.

### 4.5.1. Buckling of columns

Consider a column with some unspecified BC, under compressive load \( P \), in a critical state:

Recall that our bending analysis in Sec. 4.4, p. 12 did not account for the possibility of an axial load. We now have to revisit that analysis with the added axial load \( P \). Note that we assume that \( P || x_1 \) at all times. Also, we assume that there are no transverse loads and zero shear force in any cross section.

The moment equilibrium gives:

\[
\frac{dM}{dx_1} = P \frac{dw}{dx_1}
\]

We have to add contraction due to axial compression to \( u_1 \) in Eqn. (19):

\[
u_1 = x_2 \frac{dw}{dx_1} + \gamma x_1 + C
\]

where \( C \) is a constant determined from the BC and \( \gamma = \text{const} \) is a factor determined from the problem of uniaxial compression.

If the cross section area is \( A \) and the Young’s modulus of the material of the column is \( E \), then stress due to axial compression is \( \tau^{\text{comp}} = P/A \). Strain due to axial compression is
Displacement $u_1$ is found by integration:

$$u_1^{\text{comp}} = \int \varepsilon_1^{\text{comp}} \, dx_1 = \varepsilon_1^{\text{comp}} x_1 = \frac{P}{EA} x_1$$

So $\gamma = \frac{P}{EA}$.

Note that we use elastic superposition to solve this problem. We split the total stress/strain states into that due to bending and that due to uniaxial compression. We discuss this in more detail in Sec. 5.7.

From Eqn. (52):

$$\varepsilon_1 = x_2 \frac{d^2 w}{dx_1^2} + \gamma$$

(53)

The bending moment is:

$$M = \int_A t \, dx_2 = E \int_A \varepsilon_1 x_2 \, dA = E \int_A (x_2 \frac{d^2 w}{dx_1^2} + \gamma) x_2 \, dA = E \frac{d^2 w}{dx_1^2} \int_A x_2^2 \, dA + E \gamma \int_A x_2 \, dA$$

The first integral on the right hand side is $I$, the second moment of area. The second integral is $i$, the first moment of area. Given that the analysis is considered wrt the neutral line, $i = 0$. So the differential equation of bending is identical to Eqn. (29) from Sec. 4.4:

$$M = EI \frac{d^2 w}{dx_1^2}$$

(54)

By differentiating Eqns. (51) and (54) over $x_1$:

$$\frac{dM}{dx_1} = \frac{d}{dx_1} \left( EI \frac{d^2 w}{dx_1^2} \right) = \frac{d}{dx_1} (P dw)$$

Finally, assuming $P = \text{const}$, $E = \text{const}$ and $I = \text{const}$, one obtains a 3rd order linear ODE:

$$EI \frac{d^3 w}{dx_1^3} = P \frac{dw}{dx_1}$$

or

$$\frac{d^3 w}{dx_1^3} - \frac{P}{EI} \frac{dw}{dx_1} = 0$$

(55)

where by introducing:

$$z^2 = -\frac{P}{EI}$$

($P$ is compressive, hence negative):

$$\frac{d^3 w}{dx_1^3} + z^2 \frac{dw}{dx_1} = 0$$

(56)

the solution to which is:

$$w = C_1 \cos zx_1 + C_2 \sin zx_1 + C_3$$

(57)

Let’s find the integration constants for a simply supported column from both ends:

$$w(x_1 = 0) = 0, \quad w(x_1 = l) = 0,$$

The length of the column is $l$. There are plenty of BC we can use: $w(x_1 = 0) = 0, \quad w(x_1 = l) = 0,$

$$w = \frac{2}{3} \frac{P}{l}$$
\( w'(x_1 = l/2) = 0, \quad w''(x_1 = 0) = 0, \quad w''(x_1 = l) = 0. \)

\[ w''(x_1 = 0) = 0 \quad \Rightarrow \quad C_1 = 0 \]

\[ w(x_1 = 0) = 0 \quad \Rightarrow \quad C_1 + C_3 = 0 \]

\[ w''(x_1 = l) = 0 \quad \Rightarrow \quad C_2 \sin zl = 0 \]

\( C_2 = 0 \) is a trivial solution for a straight, unbuckled, column. This solution is of no interest to us. Hence it must be that:

\[ \sin zl = 0 \]

or

\[ zl = n\pi \]

Finally, the magnitude of the critical load is

\[ P_{\text{crit}} = EI \left( \frac{n\pi}{l} \right)^2 \]

The lowest critical load corresponds to \( n = 1 \), i.e. when the deformed shape of the column is half of the sine wave:

\[ P_{\text{lowest}}^{\text{crit}} = \pi^2 \frac{EI}{l^2} \quad (58) \]

Note that we cannot fit \( C_2 \) from the BC. This means that any \( C_2 \) fits the BC, provided it’s not too high to violate the small deflection assumption. The deflection is

\[ w = C_2 \sin \frac{n\pi x_1}{l} \quad (59) \]

(Ex. prob. 20, 21, 22, 23).

5. Three-dimensional stress/strain analysis

5.1. Motivation

We have already defined one dimensional stress vector as \( t = f/A \), Eqn. (5) in Sec. 4.1.1 on p. 3, and refined it as \( t = \lim_{\Delta A \to 0} f/\Delta A \), Eqn. (22) in Sec. 4.4 on p. 12. Why do we need anything else?

Because the above vector definitions lead to simple scalar axial stress expressions, \( t = f/A \) or \( t = \lim_{\Delta A \to 0} f/\Delta A \), only for very specific CS. We will now illustrate what happens when different CS is chosen, e.g. for a problem of a cable under tension.

The cable is loaded by force \( f \) along the axis. The cross section of the cable, cut normal to the axis, is \( S_0 \).
The stress vector acting on an element of surface normal to the axis is \( \mathbf{t}_0 = f/S_0 \). This is the left diagram above.

Consider now what happens if I choose a different CS, rotated in the plane of the drawing by angle \( \theta \). This is the middle diagram above. The cross section area has changed. It is now \( S = S_0/\cos \theta \). Hence \( S \geq S_0 \). However, from equilibrium, the force acting on this element of surface is still \( f \). Hence the stress vector is

\[
\mathbf{t} = \frac{f}{S} = \frac{f \cos \theta}{S_0} = \mathbf{t}_0 \cos \theta
\]

Clearly \( \mathbf{t} \parallel \mathbf{t}_0 \) and \( \mathbf{t} \leq \mathbf{t}_0 \).

Moreover, the stress vector in the new CS, \( \mathbf{t} \), is no longer normal to the cross section, given by the normal vector \( \mathbf{n} \). Hence we can split this vector into a normal and a shear components. This is the right diagram above. The normal stress is

\[
t^n = t \cos \theta = t_0 \cos^2 \theta
\]

The shear stress is

\[
t' = t \sin \theta = t_0 \cos \theta \sin \theta = \frac{1}{2} t_0 \sin 2\theta
\]

Note the use of the double angle in the shear stress expression. This will be significant in Sec. 5.6.

**Conclusion:** the values of the normal and the shear stresses acting on an element of surface depend on its orientation. However, the stress state is a physical property which cannot depend on the orientation of CS.

For illustration, let’s calculate the values of the normal and the shear stresses for certain values of \( \theta \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \cos \theta )</th>
<th>( \sin \theta )</th>
<th>( \cos^2 \theta )</th>
<th>( \cos \theta \sin \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \pi/6 )</td>
<td>( \sqrt{3}/2 )</td>
<td>( 1/2 )</td>
<td>( 3/4 )</td>
<td>( \sqrt{3}/4 )</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>( \sqrt{2}/2 )</td>
<td>( \sqrt{2}/2 )</td>
<td>( 1/2 )</td>
<td>( 1/2 )</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>( 1/2 )</td>
<td>( \sqrt{3}/2 )</td>
<td>( 1/4 )</td>
<td>( \sqrt{3}/4 )</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

It turns out that to fully describe the stress state at a point one has to know stress vectors acting on three elements of surface passing through that point, each with different normal. Together these 3 vectors, e.g. \( \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \), form stress tensor, which is the subject of the next chapter.
5.2. Stress

Consider a body:

The following analysis is originally due to Cauchy (1822). We will use a Cartesian CS, \( x_i \), to give coordinates to points in this body. Point \( P \) is defined by vector \( \mathbf{x}_P = x_i^p, i = 1, 2, 3 \).

Consider an imaginary planar cut across the body, with normal \( \mathbf{n} = n_i \), through point \( P \). Imagine a small area of surface, \( \Delta S \), on this plane, that includes point \( P \). Finally, imagine that \( \mathbf{f}^n = f_i^n \) is the force acting on \( \Delta S \). This is an internal force, as opposed to any surface loadings. Superscript \( n \) denotes that this is the force acting on the element with normal \( \mathbf{n} \).

The stress vector definition, introduced by Eqn. (22) in Sec. 4.4 on p. 12, is still adequate. The only minor change is the superscript \( n \), to emphasise where the force and the stress are acting.

\[
\mathbf{t}^n = \lim_{\Delta S \to 0} \frac{\mathbf{f}^n}{\Delta S} \tag{60}
\]

In general \( \mathbf{t}^n \) will depend on the position of point \( P \), and also on the orientation of \( \Delta S \), i.e. on \( \mathbf{n} \). In other words \( \mathbf{t}^n = t_i^n(\mathbf{x}_P, \mathbf{n}) \). How does this function look like? To answer this question, we will use the conservation of linear momentum law (Newton’s second law) applied to a small body of mass \( \Delta m \). Deformation changes the density and volume of the body, but the mass is maintained constant.

\[
f = \Delta m \frac{d\mathbf{v}}{dt} = \rho \Delta V \mathbf{x} \tag{61}
\]

where \( f \) is the force acting on a body moving with velocity \( \mathbf{v} \), \( t \) is time, \( \rho \) is density, \( V \) is volume, dot denotes time derivative and \( \mathbf{x} \) is acceleration.

Let’s apply this law to a pyramid (tetrahedron) built from point \( P \) with edges along the coordinate axes. \( h \) is the height of the pyramid from \( P \) to the base, along \( \mathbf{n} \). \( \mathbf{t}^n \) is the stress acting on the base.
Let $\Delta S$ be the area of ABC. The volume of pyramid PABC is $\Delta V = \frac{1}{3} h \Delta S$.

Let's now write down (and draw) all forces acting on the pyramid. We have to assume that there are forces, or stress vectors, also on the other three faces - ABP, BCP, ACP. We denote the forces as $\mathbf{f}^i, i = 1, 2, 3$, so that e.g. $\mathbf{f}^1$ is the force acting on the element of surface with normal along axis 1, i.e. BCP. The stresses on the faces are $\mathbf{t}^i$, so that e.g. $\mathbf{t}^2$ is the stress acting on the element of surface with normal along axis 2, i.e. ACP.

The force and the stress vectors can be split into components along the coordinate axes. Thus $\mathbf{f}^1 = (f_{11}^1, f_{12}^1, f_{13}^1)$, where $f_{11}^1$ is the force along direction 1 acting on the element of surface with normal along 1, $f_{12}^1$ is the force along direction 2 acting on the element of surface with normal along 1 and $f_{13}^1$ is the force along direction 3 acting on the element of surface with normal along 1. $\mathbf{t}^2 = (t_{21}^2, t_{22}^2, t_{23}^2)$.

So the superscript denotes where the stress or force is acting, and subscript denotes in which direction it acts.

We adopt the following sign convention. On planes with normals pointing in the positive coordinate direction the stresses are positive if they also point in the positive coordinate direction. On planes with normals pointing in the negative coordinate direction stresses are positive if they also point in the negative coordinate direction. For normal stresses this simply means that if they point in the same direction as the normal, then they are positive; if they point in the opposite direction to the normals, then they are negative. Let's assume positive forces for simplicity.

The force along 1 is

$$- f_1^1 - f_2^2 - f_3^3 + f_1^n = - t_1^1 \Delta S^1 - t_2^2 \Delta S^2 - t_3^3 \Delta S^3 + t_1^n \Delta S$$

where $\Delta S^1, \Delta S^2$ and $\Delta S^3$ are the respective surface areas. or, since $\Delta S^1$ is the projection of $\Delta S$ on the element of surface with normal along 1, then

$$\Delta S^1 = \Delta S |n \cdot n^1| = \Delta S n_1$$  (62)
where \( \mathbf{n} \) is the normal to \( \Delta S \). With that the force along 1 is:

\[-t_1^1 \Delta S n_1 - t_1^2 \Delta S n_2 - t_1^3 \Delta S n_3 + t_1^n \Delta S = (-t_1^1 n_1 - t_1^2 n_2 - t_1^3 n_3 + t_1^n) \Delta S\]

Also, we have to allow for the possibility of a body force, \( \mathbf{b} \), which acts on a volume, as opposed to \( \mathbf{f} \), which acts on a surface. The unit of the body force is force per unit volume. With that the total force along 1 is:

\[ (-t_1^1 n_1 - t_1^2 n_2 - t_1^3 n_3 + t_1^n) \Delta S + b_1 \Delta V \]

Finally the conservation of linear momentum along 1 looks like:

\[ (-t_1^1 n_1 - t_1^2 n_2 - t_1^3 n_3 + t_1^n) \Delta S + b_1 \Delta V = \rho \Delta V \ddot{\mathbf{x}} \]

A critical step - the linear momentum must be conserved for any arbitrary volume, including an infinitesimal volume, i.e. when \( h \to 0 \). \( \Delta V = O(h \Delta S) \) hence \( \Delta V = o(\Delta S) \), meaning that as \( \Delta S \to 0 \), \( \Delta V/\Delta S \to 0 \). In simple words \( \Delta V \) tends to zero faster than \( \Delta S \), so the \( \Delta V \) terms can be dropped, so finally the conservation of linear momentum along 1 looks like:

\[ t_1^1 n_1 + t_1^2 n_2 + t_1^3 n_3 = t_1^n \]

and similar equations for directions 2 and 3:

\[ t_2^1 n_1 + t_2^2 n_2 + t_2^3 n_3 = t_2^n \]
\[ t_3^1 n_1 + t_3^2 n_2 + t_3^3 n_3 = t_3^n \]

The last 3 equations can be written as one vector equation:

\[ \sum_{i=1}^{3} t_j^i n_i = t_j^n \]

where we used the Einstein’s summation convention which says that if an index repeats exactly twice in a term of an expression, then summation is assumed over this index. This summation index is called dummy.

By convention the 9 scalars \( t_j^i \) are written as \( \sigma_{ij} \). The classical form of the above equation is:

\[ \sigma_{ij} n_i = t_j^n \]  \hspace{1cm} (63)

or in tensor notation:

\[ \mathbf{\sigma} \cdot \mathbf{n} = \mathbf{t^n} \]  \hspace{1cm} (64)

The significance of this equation is that the stress state at any point in a body is described by 9 scalars. We therefore see that stress analysis requires mathematical tools beyond vectors. In fact the 9 scalars \( \sigma_{ij} \) form a rank 2 tensor (R2T) described in sec. 8.3 on p. 65.

Stress tensor can be shown graphically as a \( 3 \times 3 \) matrix containing the normal stresses on main diagonal.

\[ \mathbf{\sigma} = \sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \]

The symmetry of R2T means that \( \sigma_{ij} = \sigma_{ji} \), or, in tensor notation \( \mathbf{\sigma} = \mathbf{\sigma}^T \). So this means that \( \sigma_{12} = \sigma_{21} \), \( \sigma_{23} = \sigma_{32} \), \( \sigma_{31} = \sigma_{13} \), and the matrix can be written as:

\[ \mathbf{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \]

which is symmetric.

Symmetric matrices are sometimes shown with only the upper triangle. All elements below the main diagonal are replaced by the word sym or simply by blank space:
\[ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}_{\text{sym}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{22} & \sigma_{23} \\ \sigma_{33} \end{pmatrix} \]

5.2.1. Elementary cube of material

We have said that stress state is a continuous field, and hence stress is defined at a point. It is useful to imagine an infinitesimal volume of material centred at a point. We call this an elementary cube of material.

By definition, \( \sigma_{11}, \sigma_{22}, \sigma_{33} \) are called normal stresses, and all other \( \sigma_{ij}, i \neq j \) are called shear stresses. By convention, positive normal stresses are tensile and negative normal stresses are compressive. The normal stresses are shown on the elementary cube of material with arrows normal to the cube faces. The shear stresses are shown with arrows parallel to the cube faces:

In this example \( \sigma_{11} \) and \( \sigma_{31} = \sigma_{13} \) are negative. All other stress components are positive.

We will show later that the stress tensor is symmetric, i.e. \( \sigma_{ij} = \sigma_{ji} \).

(Ex. prob. 25.)

5.2.2. Conservation of linear momentum

Let’s see what the conservation of linear momentum means for an arbitrary body of volume \( V \) and surface \( S \):

\[ \int_V b_i \, dV + \int_S t_i^n \, dS = \int_V \rho \ddot{x}_i \, dV \quad (65) \]

(63) \( \rightarrow \) (65):

\[ \int_V b_i \, dV + \int_S \sigma_{ij} n_j \, dS = \int_V \rho \ddot{x}_i \, dV \]

or using the Green’s theorem,

\[ \int_V b_i \, dV + \int_V \sigma_{ij,j} \, dV = \int_V \rho \ddot{x}_i \, dV \]

where subscripts after a comma denote spatial differentiation: \( \sigma_{ij,j} = \partial \sigma_{ij} / \partial x_j \).

Further:

\[ \int_V (b_i + \sigma_{ij,j} - \rho \ddot{x}_i) \, dV = 0 \]

which must hold \( \forall V \), so the integrand is zero:

\[ \sigma_{ij,j} = -b_i + \rho \ddot{x}_i \quad (66) \]

or in tensor notation:

\[ \nabla \cdot \sigma = -\mathbf{b} + \rho \ddot{\mathbf{x}} \quad (67) \]
If the acceleration and body forces are small and can be neglected, then the equation is particularly simple.

\[ \sigma_{ij,j} = 0 \]  

or in tensor notation:

\[ \nabla \cdot \sigma = 0 \]

These equilibrium equations are one of the few fundamental building blocks of the continuum solid mechanics.

(Ex. probs. 26, 27)

### 5.2.3. Conservation of angular momentum

First read sec. 5.3 until you understand the alternating tensor. The conservation of linear momentum law effectively says: force changes velocity. Similarly, the conservation of angular momentum law says: moment of force changes angular momentum.

For a body of mass \( m \) with velocity \( \mathbf{v} \), the angular momentum is

\[ \mathbf{L} = m\mathbf{v} \times \mathbf{x} \]

where \( \mathbf{x} \) is the vector from the axis of rotation to the centre of mass of the body and \( \times \) is the cross product of velocity and the position vectors. Note that \( \mathbf{L} \) is directed along the axis of rotation: \( \mathbf{L} \perp \mathbf{v} \), \( \mathbf{L} \perp \mathbf{x} \). If the force \( \mathbf{f} \) is applied to a deformable body of unchanging mass, then the conservation of the angular momentum law is:

\[ md(\mathbf{v} \times \mathbf{x}) = \mathbf{f} \times \mathbf{x} \]

or

\[ m \frac{d}{dt}(\mathbf{v} \times \mathbf{x}) = \mathbf{f} \times \mathbf{x} \]

For a deformable body we have to apply this law to every element of mass \( dm = \rho dV \).

We now switch to index notation, in which \( \mathbf{v} \times \mathbf{x} \) is expressed as \( e_{ijk} v_j x_k \), where \( e_{ijk} \) is the alternating tensor explained in Sec. 5.3. With that Eqn. (70) can be rewritten as

\[ dL_i = \rho e_{ijk} v_j x_k dV \]

or, for the whole body:

\[ L_i = \int_v \rho e_{ijk} v_j x_k dV \]

We need to take into account the moments of the body and the surface forces. So the conservation law is:

\[ \int_v \frac{d}{dt}(e_{ijk} v_j x_k) dV = \int_v e_{ijk} b_j x_k dV + \int_S e_{ijk} t_j x_k dS \]

(72)

Using Eqn. (63) and the Green’s theorem the last integral in Eqn. (72) can be converted to a volume integral:

\[ \int_S e_{ijk} t_j x_k dS = \int_S e_{ijk} \sigma_{jp} n_p x_k dV = \int_v e_{ijk}(\sigma_{jp} x_k)_p dV = \int_v e_{ijk}(\sigma_{jp} x_k + \sigma_{jp} x_k,p) dV \]

it is easy to show that \( \sigma_{jp} x_k,p = \sigma_{jk} \). Using the equilibrium equations, Eqn. (66):

\[ = \int_v e_{ijk}(-b_j x_k + \rho \dot{x}_j x_k + \sigma_{jk}) dV \]

Expanding the first integral in Eqn. (72):

\[ \int_v \frac{d}{dt}(e_{ijk} v_j x_k) dV = \int_v e_{ijk}(\rho \dot{v}_j x_k + \rho v_j \dot{x}_k) dV \]

So that Eqn. (72) can be rewritten as:
\[ \int_V e_{ijk}(\rho v_j x_k + \rho v_j x_k - b_j x_k + b_j x_k - \rho \bar{v}_j x_k - \sigma_{jk})dV = 0 \]

This equation must be valid for any arbitrary volume, hence the integrand is zero. After cancelling terms we have:

\[ e_{ijk}(\rho v_j v_k - \sigma_{jk}) = 0 \]

It is easy to show that \( e_{ijk} v_j v_k = 0 \), hence finally we have

\[ e_{ijk} \sigma_{jk} = 0 \]

and this means that \( \sigma \) is symmetric:

\[ \sigma_{jk} = \sigma_{kj} \] (73)

or, in tensor notation:

\[ \sigma = \sigma^T \] (74)

(Ex. probs. 37, 38, 39, 40, 41, 42, 43).

### 5.3. Tensors

Imagine two orthonormal CS: \( x_i \) and \( x_i' \), with the common origin. In other words, the primed, \( ' \), CS is rotated with respect to the original, unprimed, CS. We call any such rotation a coordinate transformation or CT for short. So any vector \( \bar{a} \) will have components \( a_i = (a_1, a_2, a_3) \) in the original CS and \( a_i' = (a_1', a_2', a_3') \) in the new, rotated, CS. The vector itself does not change with any CT. What is changing are its components. We say that a vector is an object that is invariant to CT.

By definition, scalars \( A_{ij}, i, j = 1 \cdots N \), where \( N \) is the dimensionality of space, form a rank 2 tensor or R2T, if for any 2 arbitrary vectors, \( b_j \) and \( c_j \), the product: \( A_{ij} b_j c_j \) is invariant to CT:

\[ \text{if } \forall b_j, c_j : A_{ij} b_j c_j = A_{ij'} b_j' c_j' \cdot \text{ A is R2T} \] (75)

(Ex. prob. 28)

In 2D space R2T has \( 2^2 = 4 \) components. In 3D space R2T has \( 3^2 = 9 \) components. In general R2T tensor has \( N^2 \) components in \( N \)-dimensional space.

R2T transforms a vector into another vector:

\[ c_i = A_{ij} b_j \quad \text{or} \quad \mathbf{c} = \mathbf{A} \cdot \mathbf{b} \] (76)

(Ex. prob. 29)

Rotation tensor, \( R_{ij} = \mathbf{R} \), is the key. \( \mathbf{R} \) rotates vector \( \mathbf{a} \), leaving its magnitude intact. The rotated vector is denoted \( \mathbf{a}' \). We use \( ' \), to denote a vector in a new, rotated, CS. So the basis vectors are rotated from \( \mathbf{e}^i \) into \( \mathbf{e}'^i \): It is easy to show that for a CT each \( ij \) component of \( \mathbf{R} \) is \( \cos \angle(\mathbf{e}'^i, \mathbf{e}'^j) \). Finally \( \mathbf{R} \) is orthogonal: \( \mathbf{R}^{-1} = \mathbf{R}^T \), where superscript \(-1\) means the inverse and superscript \( T \) denotes a transpose of a R2T, and \( \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I} = \delta_{ij} \), where \( \mathbf{I} = \delta_{ij} \) is the rank 2 identity tensor, also called Kronecker delta tensor. Its matrix analogue is the unit matrix.

If \( \mathbf{R} \) rotates vector \( \mathbf{a} \) into \( \mathbf{a}' \), then \( \mathbf{R}^T \) rotates \( \mathbf{a}' \) back into \( \mathbf{a} \). Remember that the vector \( \bar{a} \) does not change! What is changing are its components in two different CS.

(Ex. probs. 30, 31, 32)

It is also easy to show that components of R2T change like this:

\[ T'_{ij} = R_{ip} R_{jq} T_{pq} \quad \text{or} \quad \mathbf{T}' = \mathbf{R} \cdot \mathbf{T} \cdot \mathbf{R}^T \] (77)

which can be taken as an alternative definition of R2T. To enable manipulations with tensors using matrices, one must have identical last subscript of the first tensor and the first subscript of the second tensor, i.e. \( \mathbf{A} \cdot \mathbf{B} = A_{pq} B_{qr} \), \( \mathbf{B} \cdot \mathbf{A} = B_{pq} A_{qr} \), and

\[ (\mathbf{B} \cdot \mathbf{A})^T = \mathbf{A}^T \cdot \mathbf{B}^T = A_{pq} B_{qr}^T = (B_{pq} A_{qr})^T \]
Similar to Eqns. (75) and (77) we can define tensors of arbitrary rank. For example:

\[
\text{if } \forall b_i : a_i a_i = a_i' a_i' : a_i \text{ is rank 1 tensor, i.e. vector (78)}
\]

\[
\text{if } \forall b_i, c_i, d_i : M_{ijk} b_i c_j d_k = M_{ijk}' b_i' c_j' d_k' : M_{ijk} \text{ is rank 3 tensor (79)}
\]

\[
\text{if } \forall b_i, c_i, d_i, e_i : Q_{pqrs} b_p c_q d_r e_s = Q_{pqrs}' b_p' c_q' d_r' e_s' : Q_{pqrs} \text{ is rank 4 tensor (80)}
\]

from where we can write down how components of tensors of various ranks change with CT. For ranks 1, 3 and 4 these transformations respectively are:

\[
a_i' = R_{ij} a_j \quad \text{(81)}
\]

\[
M_{ijk}' = R_{ij} R_{jk} M_{pqrs} \quad \text{(82)}
\]

\[
Q_{ijkl}' = R_{ip} R_{jq} R_{kr} R_{ls} Q_{pqrs} \quad \text{(83)}
\]

Two tensors of the same rank are equal iff all corresponding components are equal. For example, rank 4 tensors \( B \) and \( D \), are equal iff

\[
B_{1111} = D_{1111}, \quad B_{2111} = D_{2111}, \quad B_{3111} = D_{3111}, \ldots, \quad B_{3333} = D_{3333}.
\]

A R2T \( A \) is symmetric iff \( A = A^T \), or, in index notation, iff \( A_{ij} = A_{ji} \).

A R2T is skew-symmetric or anti-symmetric iff \( A = -A^T \), or, in index notation, iff \( A_{ij} = -A_{ji} \).

Any two tensors of the same rank can be added or subtracted. The result is a tensor of the same rank, e.g. \( A_{ij} + B_{ij} = C_{ij}, \quad M_{pqrs} - N_{pqrs} = Z_{pqrs} \).

Any R2T can be represented as sum of a symmetric and a anti-symmetric tensors:

\( A_{ij} = \text{sym}(A_{ij}) + \text{asym}(A_{ij}) \).

(Ex. prob. 34, 35, 36).

A helpful artificial rank 3 tensor is the alternating tensor, \( e_{ijk} \), also called permutation or Levi-Civita tensor in some books. By definition:

\[
e_{ijk} = \begin{cases} 
1, & ijk = 123, 231, 312 \\
-1, & ijk = 132, 321, 213 \\
0, & \text{otherwise}
\end{cases} \quad \text{(84)}
\]

\( e_{ijk} \) is useful for writing down cross products and determinants.

(Ex. pros. 44, 45, 46, 47, 48.)

5.3.1. Eigenvalue / vector

Let \( T \) be a symmetric R2T. The eigenvalue/eigenvector problem is to find 3 scalar values \( \lambda_\alpha \) and 3 vectors \( x^\alpha, \alpha = 1 \ldots 3 \), so that:

\[
T \cdot x^\alpha = \lambda_\alpha x^\alpha \quad \text{(85)}
\]

Think about the meaning of Eqn. (85). We said before that a R2T transforms a vector into another vector. Since \( \lambda_\alpha x^\alpha \| x \), we see that \( x^\alpha \) are chosen so that the action of \( T \) on them does not rotate but scales them. \( \lambda_\alpha \) are called the principal values or eigenvalues and the vectors \( x^\alpha \) are called the principal vectors or eigenvectors.

Eqn. (85) has non-trivial solutions iff

\[
\det(T - \lambda I) = 0 \quad \text{(86)}
\]

which is called the characteristic equation. In 3D space this is a cubic equation for \( \lambda \):

\[
-\lambda^3 + I^T \lambda^2 + \frac{1}{2} (II^T - (I^T)^2) \lambda + III^T = 0 \quad \text{(87)}
\]

where \( I^T, II^T, III^T \) are called the first, second and third invariants respectively. By definition
where \( \text{tr} \) is called the trace of \( R^2 T \). This operator is defined only for \( R^2 T \). It is the sum of all diagonal components.

Some texts define the second invariant as \( \frac{1}{2} (T : T - (\text{tr} T)^2) \). Ex. prob. 51 shows why.

If \( T \) is symmetric then it is possible to prove that all 3 eigenvalues are real. After Eqn. (86) is solved, each \( \lambda_\alpha \) is substituted back into Eqn. (85) and each \( x^\alpha \) is found. By convention:

\[
\lambda_1 \geq \lambda_2 \geq \lambda_3
\]  

(91)

If only one principal value is non-zero, and the other two values are zeros, the tensor is called uniaxial.

If two principal values are non-zero, and one is zero, the tensor is called biaxial. If the two non-zero values are equal, the tensor is called equi-biaxial.

If all three principal values are non-zero, the tensor is called triaxial. If, in addition, all three values are identical, the tensor is called hydrostatic or equi-triaxial.

(Ex. probs. 49, 50, 51, 52, 53, 54, 55, 56, 57).

5.4. Strain

When stresses are applied to a body it deforms. Solid mechanics is the mechanics of deformable bodies. If the bodies do not deform, then the subject is called mechanics of rigid bodies, which is a different subject completely. Analysis of deformation and motion of solids, sometimes also called kinematics is the subject of this section.

In the analysis of motion one must distinguish the original (also called undeformed or reference) configuration from current (also called deformed) configuration. We use capital letter to refer to the original configuration, and small to refer to the deformed configuration, wherever possible.

Consider point \( P \) with coordinates \( X \) in the original configuration, Imagine that after loading point \( P \) displaced by distance \( u \) and now has coordinates \( x \), and we call it \( p \). Consider another point \( Q \) some short distance \( dX \) from \( P \) in the original configuration. Its location after deformation is labelled \( q \), a distance \( dx \) from \( p \). Displacement vector connecting \( Q \) and \( q \) is \( u + du \).

One of the major assumptions of the continuum mechanics is that in general the motion of a body under load can be described by continuous, smooth, and therefore differentiable functions:
\[ x = x(X, t) \]  

(92)

where \( t \) is time or some other parameter. The motion that is very slow is said to be *quasi-static*.

These functions are *non-linear* in general, however their differentials are related linearly:

\[ dx = \frac{\partial x}{\partial X} dX \]  

(93)

or in index notation:

\[ dx_i = \frac{\partial x_i}{\partial X_j} dX_j \]  

(94)

By definition

\[ F = F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial x}{\partial X} \]  

(95)

is called the *deformation gradient* tensor. With that Eqn. (93) can be rewritten:

\[ dx = F \cdot dX \]  

(96)

Deformation must be *reversible*, meaning the inverse of the deformation gradient must always exist:

\[ F^{-1} = F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} = \frac{\partial X}{\partial x} \]  

(97)

The deformation gradient is a special type of R2T, called *two-point* tensor. By the definition of the inverse of R2T

\[ F^{-1} \cdot F = F_{ij}^{-1} F_{jk} = F \cdot F^{-1} = F_{ij} F_{jk}^{-1} = I = \delta_{ik} \]  

(98)

The square of the length of \( PQ \) is \( (dL)^2 = dX \cdot dX \), and of \( pq \) is \( (dl)^2 = dx \cdot dx \). The change in the square of the length, due to motion, \( \Delta L^2 \), is

\[ \Delta L^2 = dx \cdot dx - dX \cdot dX = dx_i dx_i - dX_i dX_i \]  

(99)

or using Eqn. (96):

\[ \Delta L^2 = F_{ij} dX_j F_{ik} dX_k - dX_i dX_i = F_{ij} dX_j F_{ik} dX_k - dX_j dX_k \delta_{ik} = dX_j dX_k (F_{ij} F_{ik} - \delta_{jk}) \]  

(100)

or finally

\[ \Delta L^2 = dX_j dX_k (F_{ij} F_{ik} - \delta_{jk}) = (dX \otimes dX) : (F^T \cdot F - I) \]  

(101)

A change in length is a measure of *stretch*.

By definition

\[ C = C_{jk} = F_{ij} F_{ik} = F^T \cdot F \]  

(102)

is called the *left* Cauchy-Green tensor.

Stretch is measured in units of length. What we want is a *relative dimensionless* measure of deformation, which we call *strain*. There are many different definitions of strain. We can adopt one suggested by Eqn. (101):

\[ E = \frac{1}{2} (F^T \cdot F - I) = E_{jk} = \frac{1}{2} (F_{ij} F_{ik} - \delta_{jk}) \]  

(103)

The factor \( \frac{1}{2} \) will be explained later.

Strain can be expressed via displacements as:

\[ E_{jk} = \frac{1}{2} (u_{ij,j} + u_{ji,j} + u_{ki,k} u_{kj,j}) = E = \frac{1}{2} (\nabla u + (\nabla u)^T + (\nabla u)^T \cdot (\nabla u)) \]  

(104)

If displacement gradients \( \nabla u \) are small such that their squares can be neglected: \( |(\nabla u)^T \cdot (\nabla u)| = o(|\nabla u|) \), then *small strain formulation* will result:
\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) = \varepsilon = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \]  

Strain is a symmetric R2T, hence all previous analysis regarding the properties of R2T applies to strain. In particular \( \varepsilon_{11}, \varepsilon_{22}, \) and \( \varepsilon_{33} \) are called normal strains. Positive values mean elongation and negative mean contraction. \( \varepsilon_{12}, \varepsilon_{23}, \) and \( \varepsilon_{31} \) are called shear strains. Shear strains describe change of shape, i.e. change in angle between two arbitrary straight lines before and after the motion.

The principal strains are:

\[ \varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3 \]  

The first invariant of strain tensor, \( I^\varepsilon = \varepsilon_{ii} \), quantifies volumetric strain. This is a measure of change in volume for a unit cube of materials - if \( I^\varepsilon > 0 \), then the volume is increasing; if \( I^\varepsilon < 0 \) then the volume is decreasing. If \( I^\varepsilon = 0 \), then the strain is called incompressible. If the mass conservation is in force, then similar conclusions can be made about density, e.g. \( I^\varepsilon > 0 \) would mean decreasing density.

(Ex. probs. 58, 59, 60, 61, 62, 63, 64).

5.5. Maximum shear value

Now we understand that stress and strain states are expressed via symmetric R2T. Therefore exactly the same tensor manipulations can be done with both stress and strain. In particular it’s easy to show that the maximum shear stress, \( \tau_{\text{max}} \), and the maximum shear strain, \( \gamma_{\text{max}} \), have identical expressions:

\[ \tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} ; \quad \gamma_{\text{max}} = \frac{\varepsilon_1 - \varepsilon_3}{2} \]  

where \( \sigma_1, \sigma_3, \varepsilon_1 \) and \( \varepsilon_3 \) are the maximum and the minimum principal values of stress and strain respectively.

The maximum shear exists on planes with the normal at \( 45^\circ \) to the principal directions 1 and 3.

The above expressions show that in hydrostatic cases, i.e. when the maximum and the minimum principal values are equal, there is zero shear everywhere.

(Ex. pros. 65, 66).

5.6. Mohr’s diagram

The Mohr’s diagram, also called the Mohr’s circle, is a 2D graphical representation of a symmetric R2T, \( T_{ij} \). Consider that such tensor has principal values \( T_1, T_2 \) and \( T_3 \). Then the Mohr’s diagram is:

\[ \text{Diagram showing Mohr's circle for stress or strain tensor.} \]
The axes are the normal and the shear values. The diagram consists of three circles centred on the normal axis. The radii of the circles are equal to the maximum shear values in each principal plane. The circles intersect the normal axis at principal values. The admissible values are within the large circle, but outside of the two smaller circles. Note that the diagram is symmetric wrt the normal axis, so often only a half is drawn, with positive shear values.

The Mohr’s diagram is only useful for CS in which at least one coordinate axis is the principal direction. This means that only stress states where at least one normal stress is the principal stress can be represented on the diagram. In other words rotations about only a single coordinate axis, from the principal CS, can be visualised with the Mohr’s diagram. Here is an example.

We start from the principal orientations (left). Imagine that we start rotating the elementary cube of material by angle $\alpha$ about axis 3 (middle). Shear component $T_{12}$ appears and grows to the maximum value at $\alpha = \pi/4$ (right). The corresponding matrix representation is shown below each diagram.

Note that the Mohr’s diagram can represent at most 4 components of R2T. As long as the other components are zero, the representation in meaningful. This clearly shows the limitations of the Mohr’s visual aid.

If we continue rotating our cube further about axis 3, then the shear value $T_{12}$ starts decreasing (left and middle) until it vanishes again at $\alpha = \pi/2$, and we are back to the principal CS (right).

Conclusion: Mohr’s diagram usefulness is limited to only the most simple stress states. We do not talk about it further.

(Ex. probs. 67, 68, 69).

5.7. Elasticity

So far we have defined two symmetric R2T - stress, $\sigma$, and strain, $\varepsilon$, which describe the states of loading and of deformation respectively, at every point in the material. This section explains how the two tensors are related to each other.

Up to now the theory we have been developing was almost exclusively mathematical. The only bits of physics we have used were the fundamental conservation laws. However, we said absolutely nothing about material behaviour. It turns out that material behaviour is indeed the link between the strain and the stress tensors.
Mathematically material behaviour, sometimes also called constitutive behaviour or constitutive model is just some tensor-valued function linking stress to strain:

\[ \sigma = f(\varepsilon) \quad ; \quad \varepsilon = g(\sigma) \]  

(108)

There are probably infinitely many of such functions. But do any of them describe a real material?

The simplest solid material behaviour that we know of is called elasticity. There are many ways elasticity can be defined.

\[ \sigma = C: \varepsilon \]  

(109)

where \( C = C_{\mu\nu\rho\sigma} = \text{const} \) is a rank 4 tensor, is one such definition. \( C \) is called the elastic or the stiffness tensor.

Another definition is to say that if strains are proportional to stresses, then such material is elastic. This definition is, of course, closest to that discovered by Robert Hooke in 1660. The word "discovered" is important. It's one thing to have a mathematical description, but quite another to confirm that some real materials, under some conditions, obey this mathematical description.

The two above definitions are strictly speaking restricted to a linear elastic behaviour.

Yet another, a more general, definition of elasticity, including non-linear elasticity, is to postulate the existence of an elastic potential, \( W \), such that

\[ \sigma = \frac{\partial W}{\partial \varepsilon} = \frac{\partial W}{\partial \varepsilon^{\mu\nu}} = \sigma^{\mu\nu} \]  

(110)

or equally

\[ \varepsilon = \frac{\partial W}{\partial \sigma} = \frac{\partial W}{\partial \sigma^{\mu\nu}} = \varepsilon^{\mu\nu} \]  

(111)

Other definitions of elasticity can be given based on the key properties of this behaviour.

Elastic deformation is recoverable or reversible, meaning that if loads are applied to a body and then removed, then the body returns to the initial configuration.

Elastic deformation is path independent, meaning that stress at the end of the deformation depends only on strain at the end of the deformation. The path by which this state was reached is immaterial.

Elastic behaviour is conservative, meaning that the energy is conserved. If some amount of external work was needed to deform a body, then exactly the same amount of energy will be recovered when external loading is removed.

Elastic deformation is lossless, meaning there are no energy losses during a cycle of elastic loading/unloading.

All above characteristics of real elastic behaviour put certain constraints on the type of a mathematical functions describing elasticity in Eqn. (108). These must be single-valued, meaning that only a single stress tensor corresponds to each strain tensor, and vice versa. The functions must be reversible, meaning that if it possible to calculate stress from strain, then it must also be possible to calculate strain from stress. All these requirements are satisfied by our first definition, Eqn. (109). In particular inverting Eqn. (109) one can write:

\[ \varepsilon = S: \sigma \]  

(112)

where \( S = S_{\mu\nu\rho\sigma} = \text{const} \) is a rank 4 tensor called the compliance tensor. Clearly \( S \) is the tensor inverse of \( C \). We will show later what it means.

The following analysis can be made with either definition of Eqn. (109) or that of Eqn. (112). Let's stick with Eqn. (109).

\( C \) has \( 3^4 = 81 \) components (number of spatial dimensions to the power of rank). However, only 21 of those are independent in the most general case, due to symmetries of the strain and the stress tensors and due to the existence of the elastic potential:
In the most general case Eqn. (109) can be written explicitly in matrix form as:

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{31}
\end{pmatrix}
= 
\begin{pmatrix}
C_{1111} & C_{1122} & C_{1133} & 2C_{1112} & 2C_{1123} & 2C_{1131} \\
C_{2222} & C_{2233} & 2C_{2212} & 2C_{2223} & 2C_{2231} \\
C_{3333} & 2C_{3312} & 2C_{3322} & 2C_{3331} \\
C_{1211} & C_{1222} & C_{1233} & 2C_{1212} & 2C_{1223} & 2C_{1231} \\
C_{2311} & C_{2322} & C_{2333} & 2C_{2312} & 2C_{2323} & 2C_{2331} \\
C_{3111} & C_{3122} & C_{3133} & sym & sym & sym
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{23} \\
\varepsilon_{31}
\end{pmatrix}
\]  

or

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{31}
\end{pmatrix}
= 
\begin{pmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\
C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\
C_{3333} & C_{3312} & C_{3322} & C_{3331} \\
C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} \\
C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} \\
C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
2\varepsilon_{12} \\
2\varepsilon_{23} \\
2\varepsilon_{31}
\end{pmatrix}
\]  

Both forms can be used for calculation, but neither form can be rotated. This is because the tensor nature of the Eqn. (109) is lost after the introduction of factors 2.

It is possible to show that the required number of independent components in \( C \) can be reduced further if material symmetry is exploited. In the simplest case there are only 2 independent components in \( C \).

The main characteristic of this special case is that material properties are the same in all directions. Materials possessing this property are called isotropic. Isotropic materials are a tiny minority of materials found in nature, but due to simplicity of their analysis, they form the majority of man-made materials. All other materials are called anisotropic. The vast majority of pure solids (single crystals), minerals, and composite materials are anisotropic, as the table below shows.

<table>
<thead>
<tr>
<th>Symmetry type</th>
<th>Name</th>
<th>Constants</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>no symm. plane</td>
<td>triclinic</td>
<td>21</td>
<td>anorthosite, turquoise</td>
</tr>
<tr>
<td>1 plane</td>
<td>monoclinic</td>
<td>13</td>
<td>orthoclase (KAlSi₃O₈), igneous rocks</td>
</tr>
<tr>
<td>3 planes</td>
<td>orthotropic, orthorombic</td>
<td>9</td>
<td>Ga, U, rolled plate, wood, some composites</td>
</tr>
<tr>
<td>stretched cubic</td>
<td>tetragonal, trigonal or rhombohedral</td>
<td>6 or 7</td>
<td>7: dolomite (CaMg(CO₃)₂); 6: In, Sn, As, Bi, Hg, Se, Te, quartz (SiO₂)</td>
</tr>
<tr>
<td>1 plane, 1 axis</td>
<td>hexagonal</td>
<td>5</td>
<td>Be, Cd, Zn, Zr, carbon fibre composites</td>
</tr>
<tr>
<td>3 axes</td>
<td>cubic or isometric</td>
<td>3</td>
<td>Fe, Po, Al, Cr, Cu</td>
</tr>
<tr>
<td>point of symm.</td>
<td>isotropic</td>
<td>2</td>
<td>normalised steels, Al alloys</td>
</tr>
</tbody>
</table>

The topic of anisotropy is beyond the scope of this course. From now on we restrict the course to linear isotropic elasticity only. In this case \( C \) is vastly simplified:

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{31}
\end{pmatrix}
= 
\begin{pmatrix}
2\mu + \lambda & \lambda & 0 & 0 & 0 & \varepsilon_{11} \\
\lambda & 2\mu + \lambda & 0 & 0 & 0 & \varepsilon_{22} \\
0 & \lambda & 2\mu + \lambda & 0 & 0 & \varepsilon_{33} \\
0 & 0 & \lambda & 2\mu & 0 & \varepsilon_{12} \\
0 & 0 & 0 & \lambda & 2\mu & \varepsilon_{23} \\
0 & 0 & 0 & 0 & \lambda & \varepsilon_{31}
\end{pmatrix}
\]  

where \( \lambda \) and \( \mu \) are called the Lamé elastic constants. The tensor form of linear isotropic elasticity is:
\[ \sigma = 2\mu \epsilon + \lambda \text{tr} \epsilon \mathbf{I} \quad ; \quad \sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij} \]  

(117)

So that it’s easy to see that

\[ C_{ijkl} = 2\mu \delta_{ik} \delta_{jl} + \lambda \delta_{ij} \delta_{kl} \]  

(118)

The inverse of a rank 4 tensor is defined as follows.

\[ C^{-1} \text{ is an inverse of } C \leftrightarrow C \cdot C^{-1} = I \quad ; \quad C_{ijkl} C^{-1}_{klmn} = I_{jmn} \]  

(119)

where by definition

\[ I = l_{ijkl} = \delta_{ik} \delta_{jl} \]  

(120)

is called rank 4 identity tensor. It’s meaning is clear from Eqn. (116) - its main diagonal components are 1, and all other components are 0.

So the fact that \( S \) is an inverse of \( C \) means that

\[ C : S = S : C = I \quad ; \quad C_{ijkl} S_{klmn} = S_{ijkl} C_{klmn} = I_{jmn} \]  

(121)

Elastic law must be valid for any CS:

\[ \sigma' = C' : \epsilon' \]  

(122)

where \( C' \) is the elastic tensor in a rotated CS. It rotates as any other rank 4 tensor:

\[ C_{ijkl}' = R_{im} R_{jn} R_{ko} R_{lp} C_{mnop} \]  

(123)

However, it is possible to prove that isotropic elastic tensor is the same in any CS:

\[ C_{ijkl}' = C_{ijkl} \]  

(124)

Indeed, that is was the term “isotropic" mean.

Different applications of elasticity suggest different optimal pairs of isotropic elastic constants. Thus the Lamé constants can be recast as \( E \), the Young’s modulus, and \( \nu \), the Poisson’s ratio, or as \( K \), the bulk or compression modulus, and \( G \), the shear modulus. These are inter-related as follows:

\[ E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad ; \quad \nu = \frac{\lambda}{2(\lambda + \mu)} \]  

(125)

\[ \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad ; \quad \mu = \frac{E}{2(1 + \nu)} \]  

(126)

\[ K = \lambda + \frac{2}{3} \mu = \frac{E}{3(1 - 2\nu)} \quad ; \quad G = \mu \]  

(127)

One remarkable property of any elastic behaviour can be drawn out if one considers Eqn. (109) for two different strain states, \( \epsilon^A \) and \( \epsilon^B \). Each strain tensor has a corresponding stress state: \( \sigma^A = C : \epsilon^A \) and \( \sigma^B = C : \epsilon^B \). Imagine a composite strain state: \( \epsilon^C = \epsilon^A + \epsilon^B \). It immediately follows that its corresponding stress state is

\[ \sigma^C = C : \epsilon^C = C : (\epsilon^A + \epsilon^B) = C : \epsilon^A + C : \epsilon^B = \sigma^A + \sigma^B \]

So that the stress state resulting from a composite deformation \( A \) followed by \( B \) is a sum or superposition of stress states resulting from deformations \( A \) an \( B \). This fact is of great importance for the solution of elastic problems. It means that complex loadings can be represented as a superposition of multiple simple loadings. When the elastic problem for each simple loading has been solved, then the complete solution is obtained by the superposition of individual simple solutions.

(Ex. probs. 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81.)
5.8. Solving solid mechanics problems

A constitutive model, of which elasticity is the simplest example, completes a set of fundamental concepts and equations required to pose and solve a solid mechanics problem.

The fundamental quantities are
- $\mathbf{x} = x_i$ - the material point position vector
- $\mathbf{t^n} = t^n_i$ - the stress vector on an element of surface with normal $\mathbf{n} = n_i$
- $\mathbf{b} = b_i$ - the body force (force per unit volume) field
- $\rho$ - the scalar density field

With these concepts we have derived the Cauchy stress tensor, Eqn. (63), the small strain tensor, Eqn. (105), and the equations of equilibrium, Eqn. (67).

BC can be of two types. The first type is prescribed traction (loading). Let $\Gamma$ denote the whole of the boundary of a body. Then a traction BC is a constraint where on part of the boundary $\Gamma_t$ traction $\mathbf{t^n}$ is prescribed as $\mathbf{t^n} = \mathbf{t^n}$. A displacement BC is a constraint where on $\Gamma_u$ displacement $\mathbf{u}$ is given as $\mathbf{u} = \bar{\mathbf{u}}$.

If all BC are of a displacement type and an elastic behaviour is assumed, then nothing else is needed to pose a solid mechanics problem.

$$\begin{align*}
\nabla \cdot \mathbf{\sigma} &= -\mathbf{b} + \rho \ddot{\mathbf{x}} \\
\mathbf{\sigma} &= \mathbf{C} : \mathbf{\varepsilon} \\
\mathbf{u} &= \mathbf{u} \quad \text{on} \quad \Gamma_u
\end{align*}$$

Using the elastic relations $\mathbf{\sigma}$ is expressed as functions of $\mathbf{\varepsilon}$, and hence of $\mathbf{u}$: $\mathbf{\sigma} = \mathbf{\sigma} (\mathbf{u})$. When these are inserted back into the equilibrium equations, the result is 3 PDE for 3 unknown functions $\mathbf{u}$. These equations are then solved subject to the BC. Once $\mathbf{u}$ is found, $\mathbf{\varepsilon}$ is immediately available by the differentiation of displacements, and $\mathbf{\sigma}$ is found from the elasticity relation.

If all BC are of a traction type, then the equilibrium equations can be solved for stress directly. Once stress fields are found, strain is obtained from the elastic relationship, and displacements are calculated by the integration of strain. However, the strain-displacement relationship can be thought of as 6 PDEs for 3 unknowns $\mathbf{u}$. Such system might not have a solution, unless the strain functions are compatible. The following compatibility equations are easily obtained:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}$$

and the solid mechanics problem can be formulated like this:

$$\begin{align*}
\nabla \cdot \mathbf{\sigma} &= -\mathbf{b} + \rho \ddot{\mathbf{x}} \\
\mathbf{t^n} &= \mathbf{t^n} \quad \text{on} \quad \Gamma_t \\
\mathbf{\varepsilon} &= \mathbf{S} : \mathbf{\sigma} \\
\varepsilon_{ij,kl} + \varepsilon_{kl,ij} &= \varepsilon_{ik,jl} + \varepsilon_{jl,ik}
\end{align*}$$

Finally, if both traction and displacement BC are applied, the solid mechanics problem is of mixed type, the solution method for which is a combination of solution methods for pure traction and pure displacement BC.

There is no solution to a general 3D solid mechanics problem, even for the simplest case of linear isotropic elasticity with no body forces and under quasi-static deformation (negligible rates and accelerations). Solutions are available only for specific classes of problems, typically limited by geometry or by a type of loading. Solid mechanics is still as much art as it is science.

(Ex. prob. 82, 83, 84, 85).

6. Special cases
6.1. Two-dimensional stress/strain problems

6.1.1. Plane stress

Stress state where one principal stress is zero is called plane stress.

The stress tensor, therefore, looks like this:

\[
\sigma = \begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
\text{sym} & 0 & 0
\end{pmatrix}
\]

Importantly, there is no rotation about axis 3 that can give rise to \(\sigma_3 \neq 0\). Hence one can remove from the analysis \(\sigma_{31}, \sigma_{32}\) and \(\sigma_{33}\) completely. What remains is the two-dimensional stress tensor

\[
\sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\text{sym} & \sigma_{22}
\end{pmatrix}
\]

Accordingly we can use an elementary square, not cube, to visualise a two-dimensional stress state, because nothing happens along the third direction. Here is an example of a stress state in principal (left) and some other CS (right).

Note that the corresponding strain tensor is three-dimensional:

\[
\varepsilon = \begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} & 0 \\
\varepsilon_{22} & 0 & 0 \\
\text{sym} & \varepsilon_{33}
\end{pmatrix}
\]

where \(\varepsilon_{33}\) is calculated from Eqn. (151) in ex. prob. 79:

\[
\varepsilon_{33} = -\frac{\nu}{E} (\sigma_{11} + \sigma_{22})
\]

after the two-dimensional stress-strain problem has been solved.

A typical example of plane stress state is a very thin flat film:

where the film thickness, \(t \ll L\), where \(L\) is some characteristic dimension in the plane of the film.

Let 1 and 2 be axes in the plane of the film, and axis 3 normal to it. Then if the side surfaces are traction free, and the traction is applied only on the film edges, such that \(T_3^n = 0\), then \(\sigma_{13} = \sigma_{23} = \sigma_{33} = 0\) on the free surfaces. Because the film is very thin, it is then assumed that even if \(\sigma_{33} \neq 0\) somewhere in the
interior of the film it is very small and can be neglected. Thus a plane stress state is assumed in such cases.

As a side note, any traction free surface is in plane stress state. A typical example is a roller, a long cylinder under compressive load normal to its axis.

On both free ends of the roller plane stress state is established.

Towards the symmetry plane of the roller another two-dimensional state can be assumed. Due to friction forces material there is constrained and cannot freely move in the axial direction. Hence \( \varepsilon_{33} = 0 \), and plane strain state is established.

(Ex. prob. 86, 87).

6.1.2. Plane strain

Strain state where one principal strain is zero is called plane strain. The strain tensor therefore looks like:

\[
\varepsilon = \begin{bmatrix}
\varepsilon_1 & 0 & 0 \\
0 & \varepsilon_2 & 0 \\
\text{sym} & \varepsilon_{12} & 0
\end{bmatrix}
\]

Similar to plane stress state, direction 3 becomes of no interest, and a two-dimensional strain tensor can be used:

\[
\varepsilon = \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} \\
\text{sym} & \varepsilon_{22}
\end{bmatrix}
\]

Importantly the stress tensor is three-dimensional:

\[
\sigma = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & 0 \\
\text{sym} & \sigma_{22} & 0 \\
& & \sigma_{33}
\end{bmatrix}
\]

However, \( \sigma_{33} \) is calculated from Eqn. (117):

\[
\sigma_{33} = \lambda (\varepsilon_{11} + \varepsilon_{22})
\]

after the two-dimensional stress-strain problem has been solved.

Plane strain state is typically associated with thick components, specifically if there are factors which constrain deformation in thickness direction. One of such factors could be surface friction. Collectively, the extent of such factors is called "constraint." Four point bending of slender beams is an example of low constraint geometry, where deformation in thickness direction, between the central rollers, is not constrained.
In contrast, the hot rolling press roller, discussed above, is one example of high constraint geometry. Another, similar, example is bending of non-slender beams, where height and thickness are comparable to length, \( L = H = t \).

Here the friction between the rollers and the block will constrain deformation in thickness, \( t \) direction. Hence, on the through thickness symmetry plane, plane strain state can be assumed.

Note that in this example the stress state on the front and the rear surfaces is plane stress, because those are traction free. Hence problems like these are sometimes simplified to the analysis of two extreme cases - plane stress, representative of the surface, and plane strain, representative of the symmetry plane. The stress/strain state anywhere else in this body can be assumed to lie between these two extremes.

(Ex. prob. 88).

6.1.3. Axisymmetric

If the geometry and the loading share a symmetry axis, then such problems are called axisymmetric. In this case every cross section passing through the symmetry axis is identical, and a three-dimensional problem is reduced to a two-dimensional problem of the cross section.

Consider a CS where axes 1 and 2 are in plane of a cross section. Due to symmetry, there can be no shear on planes normal to 3:

\[
\sigma_{31} = \sigma_{32} = \varepsilon_{31} = \varepsilon_{32} = 0
\]

Moreover, because all cross sections are identical, \( \sigma_{33} \) cannot depend on \( x_3 \). Hence

\[
\sigma_{33,3} = 0
\]

Likewise, there cannot be a body force, or an acceleration, or a velocity along 3. Hence the whole right hand side of Eqn. (66) is zero along 3.

With these constraints Eqn. (66) along 3 becomes identically zero, and only two equilibrium equations remain. The problem therefore becomes two-dimensional.

However, both the stress and the strain tensors are still three-dimensional:
\[
\sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & 0 \\
\sigma_{12} & \sigma_{22} & 0 \\
\text{sym} & \text{sym} & \sigma_{33}
\end{pmatrix}; \quad \varepsilon = \begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} & 0 \\
\varepsilon_{12} & \varepsilon_{22} & 0 \\
\text{sym} & \text{sym} & \varepsilon_{33}
\end{pmatrix}
\]

\(\varepsilon_{33}\) is found from another constraint. Consider a view along the symmetry axis.

Let \(AB\) be a small element of length \(x_3\). After the deformation this element becomes \(A'B'\). It also moves along 1 by \(u_1\). From similar triangles one obtains:

\[
\frac{x_3}{x_1} = \frac{x_3 + u_3}{x_1 + u_1}
\]

So that

\[
u_3 = \frac{x_3(x_1 + u_1)}{x_1} - x_3 = \frac{x_3}{x_1} u_1
\]

Thus the axisymmetry constraint means that \(u_3\) is not independent, but rather is a function of \(u_1\).

By definition

\[
\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = \frac{\partial \left( \frac{x_3}{x_1} u_1 \right)}{\partial x_3} = \frac{u_1}{x_1}
\] (131)

After the two-dimensional stress-strain problem has been solved, and \(\varepsilon_{11}\) has been found, \(u_1\) is calculated as:

\[
u_1 = \int_0^{x_1} \varepsilon_{11} dx_1
\]

with the axisymmetric BC that \(u_1(x_1 = 0) = 0\). Then \(\varepsilon_{33}\) is found from Eqn. (131). Then \(\sigma_{33}\) is found from Eqn. (117):

\[
\sigma_{33} = 2\mu \varepsilon_{33} + \lambda \text{tr} \varepsilon
\]

6.1.4. Torsion

Torsion is probably the easiest two-dimensional problem, for which a full analytical solution is readily available. Consider a straight rod of circular cross section with radius \(R\), loaded by torque \(T\) and one end, and fixed at the other.
The geometry of the problem is axisymmetric, but the loading is skew symmetric:

Hence the torsion problem is not axisymmetric.

The assumptions that make solution to this problem particularly simple are:

- Flat cross sections initially normal to axis 2 remain flat and normal to axis 2 throughout the deformation. This assumption is strictly true only for axisymmetric (circular) cross sections.
- The length of the rod does not change.
- The twist angle, $\alpha$, changes linearly with $x_2$, i.e.

$$\alpha' = \frac{d\alpha}{dx_2} = \text{const} \quad (132)$$

The first two assumptions mean that displacements along the axis are not possible:

$$u_2 = 0$$

Therefore for any cross section containing the symmetry axis, the only non-zero displacements are normal to this cross section.

Let’s choose the cross section in 23 plane. Then

$$u_3 = 0$$

To find $u_1$ let’s consider the motion of a "slice" of height $dx_2$, cut normal to the axis:

Point $B$, located distance $x_3$ from the axis of symmetry ($0 \leq x_3 \leq R$), moves to $B'$. The total displacement of point $B$ is $\mathbf{u} = (u_1, 0, 0)$. Point $A$ moves to $A'$. The total displacement of point $A$ is $\mathbf{u} = (u_1 + du_1, 0, 0)$, i.e. both points move only along 1. Angle $\angle AOC = \alpha$, and $\angle COA' = d\alpha$, so that
\[ du_1 = d\alpha x_3 \]

Eqn. (132) can be rewritten as
\[ d\alpha = \alpha' dx_2 \]
so that
\[ du_1 = \alpha' x_3 dx_2 \]

Then from Eqn. (105)
\[ \varepsilon_{12} = \frac{1}{2} \frac{\partial u_1}{\partial x_2} = \frac{\alpha' x_3}{2} \]
\[ \varepsilon_{13} = \frac{1}{2} \frac{\partial u_1}{\partial x_3} = \alpha' dx_2 \rightarrow 0 \]

All other components of the strain tensor are zero. The strain tensor is:
\[ \varepsilon = \begin{pmatrix} 0 & \varepsilon_{12} & 0 \\ \varepsilon_{12} & 0 & 0 \\ \text{sym} & 0 & 0 \end{pmatrix} \]

The strain state is pure shear, so that \( \varepsilon_1 = \varepsilon_{12}, \varepsilon_2 = 0 \) and \( \varepsilon_3 = -\varepsilon_{12} \).

The maximum strain magnitude is achieved on the outer surface of the rod, where \( x_3 = R \).

The equations of linear elasticity reduce to
\[ \sigma_{12} = 2G\varepsilon_{12} \]

and the stress tensor is:
\[ \sigma = \begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ \text{sym} & 0 & 0 \end{pmatrix} \]

Both the strain and the stress tensors can be visualised as:

```
\[ \begin{array}{c}
  2 \\
  1 \\
  3 \\
\end{array} \]
```

The conservation of angular momentum leads to
\[ T = \int_A \sigma_{12} dA x_3 \]
where \( \sigma_{12} dA \) is the elementary force and \( x_3 \) is the shoulder.
\[ \sigma_{12} = \frac{x_3}{R} \sigma_{12}^{\max} \]

where from Eqns. (133) and (134):
\[ \sigma_{12}^{\max} = 2G\varepsilon_{12}^{\max} = G \alpha' R \]
so that the torque is:
\[ T = G \alpha' \int_A x_3^2 dA \]

where \( x_3 \) is understood as the radius, because the cross section in 23 plane is chosen arbitrarily. Therefore the integral in Eqn. (136) is the second polar moment of area, defined by Eqn. (42). Finally one obtains this
expression for the torque:

\[ T = G\alpha' J \]  \hspace{1cm} (137)

from where the twist angle is found by integration:

\[ \alpha = \int x_2 \frac{T}{GJ} dx_2 \]  \hspace{1cm} (138)

Note that Eqn. (138) is valid for variable \( T, G, J \).

(Ex. prob. 89).

6.2. Application of tensor theory to properties of areas

The second moments of area form a symmetric R2T in 2D space. This means, apart from other properties, that there is always a CS such that \( I_{12} = 0 \). Other useful properties of R2T can be used to simplify the analysis of cross sections.

(Ex. prob. 90, 91, 92, 93, 94.)
7. Example problems

P1. Show that the moment of $f$ about B can be expressed as $m = fl \sin \theta$.

P2. A cylindrical column has 3 segments of length $l$, each with different radius.

Draw the axial stress profile.

P3. A conical column of length $l$ and base radius $r_0$, made of a very light material, is compressed with an axial force at the end.

Draw the axial stress and displacement profiles.

P4. A 10 m long sandstone column of 1 m diameter is lying flat on the ground. What will be its height when it’s put vertical? What will be its diameter at the bottom? Assume that sandstone density is 2,000 kg / m$^3$, the Young’s modulus is 50 GPa, the Poisson’s ratio is 0.2.

P5. A lift of mass $m = 1$ ton is travelling down with velocity $v = 1$ m / s. When the cable length is $l_0 = 5$ m, there is an emergency stop of the pulley wheel.

Calculate the maximum increase in the stress in the cable and the maximum extension of the cable due to emergency stop. Assume the cable is made of steel with the Young’s modulus of $E = 200$ GPa and the Poisson’s ratio of $\nu = 0.33$ and the initial diameter of the cable is $d_0 = 5$ mm.

P6. Calculate the vertical displacement of the force application point, assuming both ropes are made of the same material, and are of the same initial length and cross section:
P7. Force $F$ is applied to a rigid beam hanging on three identical wires (identical material, length, cross section). Find strains in each wire.

P8. Calculate the maximum deflection in three-point bending.

P9. Given $I_{11}$ and $I_{22}$, find $I_r$.

P10. Find how $I_{11}$, $I_{22}$, $I_{12}$, change with the shift of origin: $x'_j = x_j - S_j$.

P11. Find $I_{11}$, $I_{22}$, $I_{12}$ for a rectangle with sides $W$ and $H$.

P12. Find $I_{11}$, $I_{22}$, $I_{12}$ of a right angled triangle with with sides $W$ and $H$.

P13. Compare a rectangular cross section with that of a triangle obtained from this rectangle by cutting along a diagonal.

P14. Calculate the second moments of area for a circle.

P15. Calculate the second moments of area for a circular ring.

P16. Quantify the benefits of a circular ring over solid circle cross section.

P17. Find an optimum cross section with fixed area $A$, that must fit into a square box $W \times W$, where $W^2 = 4A$.

P18. Calculate the second moments for this cross section:

P19. A pipe of 20mm outer diameter and 2mm wall thickness, made from a material with $\sigma_Y = 500$MPa, is loaded in pure bending. What is the maximum bending moment it can support before yielding?

P20. Prove that Eqn. (57) is indeed a solution to Eqn. (56).

P21. Calculate the critical buckling load for a column with clamped/clamped BC.

P22. Draw buckled shapes for columns with pin/pin and clamped/clamped BC for $n = 2, 3, 4, \cdots$.

P23. Draw a stress profile across the cross section of a buckled column.
P24. For a planar element of size $S$ with normal $n$ find its projection on plane $P$ with normal $n^p$.

P25. Illustrate this stress tensor on the elementary cube of material:

$$\sigma = \begin{pmatrix} -100 & 200 & -400 \\ 300 & 500 & -600 \end{pmatrix}$$

P26. Show that $\nabla \cdot \sigma = \sigma_{ij,j}$.

P27. For a tensor of arbitrary rank, $Z$, explain the difference between $\nabla Z$ and $\nabla \cdot Z$.

P28. Prove that stress is in $\mathbb{R}^2$.

P29. In 2D space you have vector $a_i = (1, 1)$. Show how it is transformed by tensors

$$B_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

P30. Write down components of $\mathbf{R}$ for a CT consisting of rotating about 3 by $\beta$.

P31. Prove that the rotation tensor $\mathbf{R}$ is orthogonal.

P32. Show that $a = \mathbf{R}^T \cdot a'$.

P33. Prove that $\mathbf{R}^T \mathbf{T}$ changes with CT as:

$$T_{ij}' = R_{ip} R_{jq} T_{pq} \quad \text{or} \quad \mathbf{T}' = \mathbf{R} \cdot \mathbf{T} \cdot \mathbf{R}^T$$

P34. How many components do tensors of ranks 1 to 4 have in 2D and in 3D spaces?

P35. Show that any $\mathbf{R}^T$ can be represented as a sum of a symmetric and a skew-symmetric $\mathbf{R}^T$.

P36. Find sym($\mathbf{A}$) and asym($\mathbf{A}$) for this $\mathbf{R}^T$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

P37. Prove that $\sigma_{jp} x_{k,p} = \sigma_{jk}$.

P38. Prove that $e_{ijk} v_j v_k = 0$.

P39. Prove that $e_{ijk} \sigma_{jk} = 0$ means that the stress tensor is symmetric.

P40. For a body in free fall use the equilibrium equations to find the stress state.

P41. Calculate stress state in a freely standing column of constant cross section.

P42. Calculate stress state in a cylindrical rocket accelerated by a force at one end.

P43. A block of length $L$, height $H$, thickness $T$ and density $\rho$, lying on a smooth (no friction) flat surface, is pulled at one end by the weight of mass $m$ via a pulley. Find the stress state in the body.

P44. Prove that $\det \mathbf{R} = 1$.

P45. Given vectors $\mathbf{a}, \mathbf{b}$ in 3D space, write their cross product using vector components notation and index notation with $e_{ijk}$. 
P46. Prove that $e_{pq} = -e_{qp}$.

P47. Prove that $e_{ijk}$ is a rank 3 tensor.

P48. Prove that $e_{ijk}$ is isotropic.

P49. Calculate the determinant of a symmetric R2T, $T$.

P50. Show that for a symmetric R2T, $T$, $\det T = e_{ijk}T_{ij}T_{jk}T_{ki}$.

P51. Using the definitions of R2T invariants, Eqns. (88), (89), (90), expand the characteristic equation, Eqn. (86), into a cubic equation, Eqn. (87).

P52. Prove that $I^T$ is invariant to CT.

P53. Prove that $II^T$ is invariant to CT.

P54. Prove that $III^T$ is invariant to CT.

P55. Express R2T invariants via the principal values.

P56. For a uniaxial stress state, find the principal stresses and directions.

P57. Draw this stress state in the original and the principal CS:

$$\sigma = \begin{pmatrix} -100 & 200 & -200 \\ 300 & 400 & \text{sym} \\ 100 & \text{sym} & \text{sym} \end{pmatrix}$$

P58. Prove that deformation gradient is R2T.

P59. Express the strain tensor $E$ as a function of displacements $u$.

P60. Prove that strain is R2T.

P61. Analyse this motion: $x = X$, except $x_1 = X_1 + tX_2$, $t = \text{const}$.

P62. Draw the principal strains and their directions for the strain tensor from ex. prob. 61.

P63. Apply this rotation tensor to the previous example and validate that $F' = R \cdot F \cdot R^T$:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

P64. Convert the strain gauge rosette measurements into a strain tensor.

P65. Calculate the maximum shear values and orientations.

P66. Draw planes with maximum and minimum shear values for these stress and strain tensors:

$$\sigma = \begin{pmatrix} 300 & 0 & 0 \\ 50 & 0 & \text{sym} \\ \text{sym} & -500 & \text{sym} \end{pmatrix} ; \quad \varepsilon = \begin{pmatrix} 10^{-3} & 0 & 0 \\ -2 \times 10^{-3} & 0 & \text{sym} \\ \text{sym} & 10^{-3} & \varepsilon_{33} \end{pmatrix}$$

P67. Draw the Mohr’s diagram for stress state $\sigma = 0$, except $\sigma_{12} = 200\text{Mpa}$.

P68. Draw the Mohr’s diagrams for tensors in ex. prob. 66.

P69. Use the Mohr’s diagram to find the principal values of this strain tensor:

$$\varepsilon = \begin{pmatrix} 10^{-2} & 2 \times 10^{-2} & 0 \\ -3 \times 10^{-2} & 0 & \text{sym} \\ \text{sym} & \varepsilon_{33} & \text{sym} \end{pmatrix}$$

P70. Show how the symmetry of $\sigma$ and $\varepsilon$ leads to symmetries in $C$ and $S$.

P71. Show how the existence of elastic potential leads to major symmetries in $C$ and $S$.

P72. Explain factors of 2 when elastic law is written in matrix form, Eqns. (114) and (115).
P73. Explain why uniaxial tension/compression and pure shear require only a single elastic constant.
P74. Explain what’s so special about Poisson’s ratio of 0.5.
P75. Show that in anisotropic materials shear strains cause normal stresses and vice versa.
P76. Show that in anisotropic materials principal directions of stress and strain do not coincide.
P77. Show that in isotropic materials shear stresses cause only shear strains, and normal stresses cause only normal strains.
P78. Show that in isotropic materials principal directions of stress and strain coincide.
P79. Calculate $S$ starting from Eqn. (117).
P80. Give an example of loading leading to a uniaxial strain state.
P81. Explain quantitatively the difference between the uniaxial stress and uniaxial strain states.
P82. Assuming linear isotropic elasticity express the equilibrium equations via $u$.
P83. Derive the strain compatibility equation, Eqn. (129).
P84. Prove that stress states are additive.
P85. Prove that strain states are additive.
P86. Prove that for plane stress condition, no rotation in the plane of two non-zero principal stresses can give rise to stress components acting in the plane of zero principal stress.
P87. Calculate reduction in wall thickness in a spherical air balloon under pressure.
P88. Under what conditions plane stress and plane strain states are identical?
P89. Do a full stress/strain/displacement analysis of this angle bracket:

![Angle Bracket Diagram]

P90. Show that the coordinates of the centroid form a vector.
P91. Show that the first moments of area vanish for any Cartesian CS with origin at centroid.
P92. Show that the second moments of area form symmetric $R^2T$ in 2D space.
P93. For a right angled triangular cross section find the principal values and directions of the second moment of area tensor.
P94. Show that for a square cross section the second moment about any axis passing through the centroid is an invariant.
8. Solutions to example problems

S1. Refer to the drawing:

From triangle ABC: $h/l = \sin \theta$, or $h = l \sin \theta$. Then starting from $m = fh$ one obtains $m = fl \sin \theta$.

The importance of this result is that $f \sin \theta$ is the component of the force acting normal to AB. Hence the moment can be defined also as the product of the length between the point where the force is applied, A, and the point about which the moment is sought, B, by the component of the force normal to that line: $m = f \sin \theta \cdot l$.

Clearly component of the force acting along AB does not contribute to the moment.

S2. We need a free body diagram.

I choose the origin at the base of the column and direct axis $x_1$ up. I do an imaginary cut at some height $x_1$ and focus on the upper part of the column. I replace the action of the bottom part by a reaction force $R$.

The only other force acting on the column is the force of gravity, $mg$. From equilibrium

$$R = mg = \rho V g$$

where $\rho$ is the density of material and $V$ is the volume of the part of the column above the cut. Now I just need to think how to express $V$ as a function of $x_1$. Depending on where I do the cut there will be three relationships.

$$V = \begin{cases} (3l - x_1)\pi r_1^2 & x_1 \geq 2l \\ l\pi r_1^2 + (2l - x_1)\pi r_2^2 & l \leq x_1 \leq 2l \\ l\pi r_1^2 + l\pi r_2^2 + (l - x_1)\pi r_3^2 & x_1 \leq l \end{cases}$$

So the force profile has three linear segments, with gradients of each line increasing from top to bottom each lower segment of the column is heavier than the previous.
The stress $t$ is the reaction force divided over the cross section area.

$$t = \frac{R}{A} = \begin{cases} \rho g(3l-x_1) & x_1 \geq 2l \\ \rho g(l \frac{r_1^2}{r_2^2} + (2l-x_1)) & l \leq x_1 \leq 2l \\ \rho g(l \frac{r_1^2}{r_3^2} + l \frac{r_2^2}{r_3^2} + (l-x_1)) & x_1 \leq l \end{cases}$$

or, moving $l$ outside of the brackets:

$$t = \rho gl \times \begin{cases} (3 - \frac{x_1}{l}) & x_1 \geq 2l \\ \left(\frac{r_1^2}{r_2^2} + (2 - \frac{x_1}{l})\right) & l \leq x_1 \leq 2l \\ \left(\frac{r_1^2}{r_3^2} + \left(1 - \frac{x_1}{l}\right)\right) & x_1 \leq l \end{cases}$$

Note that the stress profile is discontinuous, where the cross section changes abruptly. However, the gradient of $t$ is the same in all three segments. Can you see why?

**S3.** The free body diagram is trivial.

There is a constant force $F$ in every cross section. The axial stress is negative (compressive).

$$t = -\frac{F}{A} = -\frac{F}{\pi r^2}$$

where $A$ is the cross section area and $r$ is the radius of the cone at any cross section. So the stress at the tip of the cone tends to infinity.

The strain $e_{11}$ has the same profile as the stress.

$$e_{11} = -\frac{F}{E \pi r^2}$$
The displacement \( u_1 \) is obtained by integration of \( e_{11} \) over \( x_1 \). So first we need to express \( r \) as a function of \( x_1 \). From similar triangles we obtain:

\[
\frac{r_0}{l} = \frac{r}{l - x_1} \quad \Rightarrow \quad r = \frac{r_0}{l}(l - x_1)
\]

Then

\[
e_{11} = -\frac{Fl^2}{E\pi r_0^2(l - x_1)^2}
\]

\[
u_1 = \int_0^{x_1} e_{11} \, dx_1 = -\frac{Fl^2}{E\pi r_0^2} \int_0^{x_1} \frac{1}{(l - x_1)^2} \, dx_1 = -\frac{Fl^2}{E\pi r_0^2} \times \frac{1}{l - x_1} + C
\]

\( C \) is found from the boundary conditions - when \( x_1 = 0 \) then \( u_1 = 0 \).

\[
0 = -\frac{Fl^2}{E\pi r_0^2} \times \frac{1}{l} + C \quad \Rightarrow \quad C = \frac{Fl^2}{E\pi lr_0^2}
\]

Finally

\[
u_1 = \frac{Fl^2}{E\pi r_0^2} \left( \frac{1}{l} - \frac{1}{l - x_1} \right)
\]

So \( u_1 \to -\infty \) at the tip. What is the physical interpretation of infinite stress and displacement at the tip?

**S4.** First we need a stress profile, for which we need a free body diagram.

I set the origin at the bottom of the column. I direct axis 1 (\( x \)) upward. I imaginary cut the column at some arbitrary value of \( x \) and consider the part of the column above the cut. I substitute the lower part of the column by the reaction force. The only other force acting on the upper part of the column is the force of gravity. This is the free body diagram.

The bending moment in the cross section is zero. Why?

If the mass of the upper part of the column is \( m \), then

\[
R = mg = \rho V g = \rho(h_0 - x)A_0 g
\]

where \( V \) is the volume of the upper part of column, \( \rho \) is the density of sandstone, \( A_0 \) is the original, undeformed, cross sectional area and \( h_0 \) is the original, undeformed, length (height) of the column.

The axial stress is compressive, hence negative:

\[
t = -\frac{R}{A_0} = -\frac{\rho(h_0 - x)A_0 g}{A_0} = \rho g(x - h_0)
\]

The stress is a linear function of \( x \). \( t = 0 \) at the top, where \( x = h_0 \). \( t = -\rho gh_0 \) at the bottom, where \( x = 0 \).
Note that this is exactly the same equation as for a hydrostatic pressure in a liquid.

Putting the numbers in (I convert all lengths in m):

\[ t = -2000 \times 10 \times 10 = -2 \times 10^5 \]

Since the lengths were in m, the stress is in Pa. In MPa it will be $10^{-6}$ times less, i.e. 0.2 MPa. So the maximum absolute value of axial stress, 0.2 MPa, is at the bottom of the column.

The axial strain, \( e_{11} \), is

\[ e_{11} = \frac{t}{E} = \frac{\rho g (x - h_0)}{E} \]

where \( E \) is the Young’s modulus. So axial strain also has a linear profile along \( 1 (x) \).

Displacement \( u_1 \) is the integral of strain:

\[ u_1 = \int_0^x e_{11} \, dx = \int_0^x \frac{\rho g (x - h_0)}{E} \, dx = \frac{\rho g}{E} \int_0^x (x - h_0) \, dx = \frac{\rho g}{E} \left( \frac{x^2}{2} - h_0 x \right) + C \]

where \( C \) is the integration constant found from the boundary conditions. The boundary condition is \( u_1 = 0 \) at the bottom, where \( x = 0 \). So \( C = 0 \). Finally

\[ u_1 = \frac{\rho g x}{E} \left( \frac{x}{2} - h_0 \right) \]

\( u_1 \) at the top is

\[ u_1 = -\frac{\rho g h_0^2}{E} \frac{1}{2} \]

or, substituting the numbers (I convert all units of length to m):

\[ u_1 = -\frac{2000 \times 10 \times 10^2}{50 \times 10^9} \times \frac{10^2}{2} = -0.2 \times 10^{-4} \]

This is the value in m, or –0.02 mm. The top of the column will move down (negative) 0.02 mm, so the column will be 20 µm shorter than when lying flat on the ground.

The transverse strain is

\[ e_{22} = -\nu e_{11} = \nu \frac{\rho g (h_0 - x)}{E} \]

The transverse strain is positive, i.e. tensile. This means the diameter of the column will increase. Since \( e_{22} \) depends on \( x \), the degree of change of the diameter will vary with \( x \). However, for each \( x \), \( e_{22} \) is constant across the whole cross section.

At the bottom

\[ e_{22} = \nu \frac{\rho g h_0}{E} \]

Displacement, \( u_2 \) is found by integrating \( e_{22} \) along the radius

\[ u_2 = \int_0^r \nu \frac{\rho g h_0}{E} \, dr = \nu \frac{\rho g h_0}{E} r \]

The maximum displacement is at \( r = r_0 \), the radius of the original, undeformed, column.

\[ u_2 = \nu \frac{\rho g h_0}{E} r_0 \]

Substituting the numbers in (again keeping all lengths in m):

\[ u_2 = 0.2 \times \frac{2000 \times 10 \times 10}{50 \times 10^9} \times 0.5 = 0.4 \times 10^{-6} \]

in m, or 0.4 × 10^{-3} mm or 0.4 µm. So \( \Delta d = 2u_2 = 0.8 \) µm and the diameter of the column at the bottom will be 1,000.0008 mm.
So we need to write down expressions for $K$ and $H$ for the whole body.

$$K = \frac{mv^2}{2}$$

$$H_{\text{body}} = \frac{1}{2} \varepsilon_{11} t V$$

At the moment when the lift is stationary, $H = K$, so

$$mv^2 = \varepsilon_{11} t V$$

(139)

here $V$ is the cable volume when lift has stopped moving.

Let’s first get an estimate assuming no change in volume of the cable. In this case the solution is particularly simple. Using the Hooke’s law $t = E \varepsilon_{11}$ I get from above

$$mv^2 = \frac{t^2}{E} V$$

from where

$$t_{\text{max}} = \sqrt{\frac{mE}{V}}$$

Substituting the numbers in (because of the square root, it’s easiest to convert all units of length to m. $E = 200$ GPa = $2 \times 10^{11}$ N/m$^2$, radius of the cable is $2.5 \times 10^{-3}$ m.)

$$t_{\text{max}} = 1 \times \sqrt{\frac{10^{11} \times 2 \times 10^{11}}{5 \times \pi \times (2.5 \times 10^{-3})^2}} = 1,427 \times 10^6$$

or $t_{\text{max}} = 1.427$ GPa. This stress is probably too high for most steels, so the cable will break and the lift will fall.

Now let’s take into account the volume change. Using the Hooke’s law I rewrite (139) as

$$mv^2 = E \varepsilon_{11}^2 V$$

(140)

We also need to express $V$ via $\varepsilon_{11}$.

$$V = l A = l \frac{\pi d^2}{4}$$

where $l$, $A$ and $d$ are the length, the cross section area and the diameter of the cable when the lift has stopped.

$$l = l_0 + \Delta l = l_0 + \varepsilon_{11} l_0 = l_0 (1 + \varepsilon_{11})$$

$$d = d_0 - \Delta d = d_0 + \varepsilon_{22} d_0 = d_0 (1 + \varepsilon_{22}) = d_0 (1 - \nu \varepsilon_{11})$$

where I used the definition of the Poisson’s ratio $\nu = -\varepsilon_{22}/\varepsilon_{11}$. Substituting all this back into (140) I obtain:

$$mv^2 = E \varepsilon_{11}^2 l_0 (1 + \varepsilon_{11}) \frac{\pi}{4} (d_0 (1 - \nu \varepsilon_{11}))^2$$

This is a 4th order equation for $\varepsilon_{11}$. I solve it numerically and get $\varepsilon_{11}^{\text{max}} = 7.128 \times 10^{-3}$. This strain is above 0.2%, which is the typical elastic proof strain (see the Properties of Materials course), so the cable will either deform plastically or break.
\[ t_{\text{max}} = E e_{11}^{\text{max}} = 2 \times 10^5 \times 7.12810^{-3} = 1.426 \]

So the max. stress calculated with a more accurate approach is \( t_{\text{max}} = 1.426 \) GPa, which differs by less than 1\% from that calculated with no regard to volume change.

**S6.** Make a good free body diagram. Using the symmetry of the problem, we need to draw only a single rope loaded with half the force, \( F/2 \). The axial force is

\[ R = \frac{F}{2 \cos \alpha} \]

Hence the axial stress in each rope is

\[ t = \frac{R}{A} = \frac{F}{2A \cos \alpha} \]

where \( A \) is the cross section. The axial strain in each rope is

\[ e = \frac{t}{E} = \frac{R}{EA} = \frac{F}{2EA \cos \alpha} \]

The axial displacement is

\[ u = Le = \frac{L F}{2EA \cos \alpha} \]

and the new length of each rope is

\[ L_{\text{new}} = L(1 + e) = L(1 + \frac{F}{2EA \cos \alpha}) \]

Note that when \( \alpha \to \pi/2 \) then \( R, t, e, u \to +\infty \).

The vertical component of displacement is

\[ u_v = u \cos \alpha = \frac{FL}{2EA} \]

Now we need to take the rigid body rotation into account. Refer to the diagram below.

Both ropes elongate, but the point where the force is applied stays in the middle, due to symmetry. Hence both ropes rotate about their supports to satisfy displacement compatibility. The total vertical displacement, \( d \), can be calculated from the triangles of the diagram as follows:

\[ d = B - L \cos \alpha \]

where

\[ B^2 = (L_{\text{new}})^2 - (L \sin \alpha)^2 = L^2((1 + e)^2 - \sin^2 \alpha) = L^2(cos^2 \alpha + 2e + e^2) \]

At this point let’s introduce some realistic numbers to this problem. Let’s imagine each rope is 1 m long and is made of 2 mm diameter steel wire. \( E = 200 \) GPa, \( \alpha = 45^\circ \) and \( F = 1 \) kN. Then the strain in
wires will be: \( e = 1.125 \times 10^{-3} \). This strain is below 0.2%, which is usually taken as the yield (proof) strain for steel. Hence this is a purely elastic problem.

Since \( e^2 \) is 3 orders of magnitude smaller than \( e \), we can neglect this term. Displacement, \( d \), in mm is then:

\[
d = L((\cos^2 \alpha + 2e\sqrt{3} - \cos \alpha) = 1.591
\]

The problem is immediately clear from the free body diagram of the beam - there are only two scalar equations of equilibrium which are useful, but three unknowns:

\[
R_1 + R_2 + R_3 = F \tag{141}
\]

\[
R_3 \frac{3}{2} d + R_2 \frac{d}{2} = R_1 \frac{d}{2} \tag{142}
\]

This problem cannot be solved just from the static equations of equilibrium, other information is required. This is a classical *statically indeterminate* system.

In this particular example, the extra information comes from the fact that the beam is declared as *rigid*. We will use this fact to obtain the required third equation.

Consider the beam after the deformation of the wires has occurred. Remember that the beam is rigid, i.e. it does not deform, just rotates as a rigid body:

From the triangles:

\[
\frac{u_2 - u_3}{d} = \frac{u_1 - u_2}{d}
\]

or

\[2u_2 = u_1 + u_3\]

That’s all! The rest is just algebra. The state of stress in each wire is uniaxial.

\[
t = \frac{R}{A}
\]

\[
e = \frac{t}{E}
\]

\( x_2 \) is a coordinate along the axis of each wire.

\[
u = \int_0^L e dx_2 = \frac{tL}{E} = \frac{RL}{EA}
\]

where \( A \) is the cross section, \( L \) is the length of each wire, and \( E \) is the Young’s modulus of the wire.
material.

With this the displacement compatibility equation can be rewritten as:

\[ 2R_2 = R_1 + R_3 \]  

(143)

(141)+(143):

\[ 3R_2 = F \implies R_2 = \frac{F}{3} \]

From (141):

\[ R_1 = F - R_2 - R_3 \]

From (142):

\[ R_1 = 3R_3 + R_2 \]

subtracting these two equations:

\[ R_3 = \frac{F - 2R_3}{4} = \frac{F - \frac{2}{3}F}{4} = \frac{F}{12} \]

Finally

\[ R_1 = \frac{7}{12}F ; \quad R_2 = \frac{4}{12}F ; \quad R_3 = \frac{1}{12}F \]

One can check that the force and moment equilibrium are satisfied.

After all reaction forces are known, \( t, e \) and \( u \) are immediately available from equations above.

**S8.** Three-point bending usually means a problem like this:

![Three-point bending diagram](image)

where the beam rests on two rollers and the load is applied via a third roller.

Let’s assume the load is applied at half length of the beam. Then the problem is symmetric about \( x_1 = 0 \) plane. The free body diagram and the shear force and the bending moment profiles are:

![Free body diagram](image)

\( L \) is the beam length, i.e. the distance between the bottom rollers. Note that the shear force has a jump (discontinuity) of magnitude \( P \), at a point where the load is applied via the top roller.
Need to integrate Eqn. (29) to calculate deflection:

\[
\frac{dw}{dx_1} = \int \frac{M}{IE} \, dx_1 + C
\]

where integration constant \( C \) is found from BC.

From the moment profile:

\[
M = \begin{cases} 
  P x_1, & 0 \leq x_1 \leq L/2 \\
  \frac{P(L - x_1)}{2}, & L/2 \leq x_1 \leq L
\end{cases}
\]

so for \( 0 \leq x_1 \leq L/2 \):

\[
IE \frac{dw}{dx_1} = \int_0^{x_1} \frac{P x_1}{2} \, dx_1 + C
\]

or

\[
\frac{2IE}{P} w' = \frac{x_1^2}{2} \bigg|_0^{x_1} + C = \frac{x_1^2}{2} + C
\]

where prime, \( ' \), denotes function derivative wrt to its only argument, in this case \( w' = dw/dx_1 \).

The integration constant \( C \) is found from BC. From symmetry \( w'(x_1 = L/2) = 0 \), so:

\[
0 = L^2/8 + C \quad \rightarrow \quad C = -L^2/8
\]

so that

\[
w'(0 \leq x_1 \leq L/2) = \frac{P}{2IE} \left( \frac{x_1^2}{2} - \frac{L^2}{8} \right)
\]

For \( L/2 \leq x_1 \leq L \):

\[
IE \frac{dw}{dx_1} = \int_{L/2}^{x_1} \frac{P(L - x_1)}{2} \, dx_1 + C
\]

or

\[
\frac{2IE}{P} w' = -\frac{(L - x_1)^2}{2} \bigg|_{L/2}^{x_1} + C = -\frac{(L - x_1)^2}{2} + \frac{(L - L/2)^2}{2} + C = \frac{L^2}{8} - \frac{(L - x_1)^2}{2} + C
\]

The integration constant \( C \) is found from BC. From symmetry \( w'(x_1 = L/2) = 0 \), so \( C = 0 \) and

\[
w'(L/2 \leq x_1 \leq L) = \frac{P}{2IE} \left( \frac{L^2}{8} - \frac{(L - x_1)^2}{2} \right)
\]

Let’s calculate the slope (gradient) at both ends of the beam:

\[
w'(x_1 = 0) = \frac{P}{2IE} \left( 0^2/2 - L^2/8 \right) = -\frac{1}{16} \frac{PL^2}{IE}
\]

\[
w'(x_1 = L) = \frac{P}{2IE} \left( (L^2/8) - (L - L)^2/2 \right) = \frac{1}{16} \frac{PL^2}{IE}
\]

The values are equal in magnitude and of opposite signs of course, due to symmetry. The slope profile looks like this:
To obtain deflection, need to integrate slope:

\[ w = \int w' \, dx_1 \]

For \(0 \leq x_1 \leq L/2\):

\[
 w = \frac{P}{2EI} \int_0^{x_1} \left(\frac{x_1^2}{2} - \frac{L^2}{8} \right) \, dx_1 = \frac{P}{2EI} \left(\frac{x_1^3}{6} - \frac{L^2}{8} x_1\right)\bigg|_0^{x_1} + D = \frac{P}{2EI} \left(\frac{x_1^3}{6} - \frac{L^2}{8} x_1\right) + D
\]

The BC for this case is \(w(x_1 = 0) = 0\), hence \(D = 0\) and finally:

\[
 w(0 \leq x_1 \leq L/2) = \frac{P}{2EI} \left(\frac{x_1^3}{6} - \frac{L^2}{8} x_1\right)
\]

For \(L/2 \leq x_1 \leq L\):

\[
 w = \frac{P}{2EI} \int_{L/2}^{x_1} \left(\frac{L^2}{8} - \frac{(L-x_1)^2}{2}\right) \, dx_1 = \frac{P}{2EI} \left(\frac{L^2}{8} x_1 + \frac{(L-x_1)^3}{6}\right)\bigg|_{L/2}^{x_1} + D
\]

\[
 = \frac{P}{2EI} \left(\frac{L^2}{8} x_1 + \frac{(L-x_1)^3}{6} - \frac{L^3}{16} - \frac{(L-L/2)^3}{6}\right) + D = \frac{P}{2EI} \left(\frac{L^2}{8} x_1 + \frac{(L-x_1)^3}{6} - \frac{L^3}{12}\right) + D
\]

The BC for this case is \(w(x_1 = L) = 0\), hence:

\[
 0 = \frac{P}{2EI} \left(\frac{L^3}{8} - \frac{L^3}{12}\right) + D
\]

so that

\[
 D = -\frac{PL^3}{48EI}
\]

and finally:

\[
 w(L/2 \leq x_1 \leq L) = \frac{P}{2EI} \left(\frac{L^2}{8} x_1 + \frac{(L-x_1)^3}{6} - \frac{L^3}{12} - \frac{L^3}{24}\right) = \frac{P}{2EI} \left(\frac{L^2}{8} x_1 + \frac{(L-x_1)^3}{6} - \frac{L^3}{8}\right)
\]

One can than deploy the extremum finding tools of differential calculus. However, due to symmetry, it is clear that the maximum deflection is in the middle, at \(x_1 = L/2\). We can satisfy ourselves with checking that the deflection expressions from the left and from the right side of the beam match. From the left:

\[
 w_{\text{max}}^L(x_1 = L/2) = \frac{P}{2EI} \left(\frac{L^3}{48} - \frac{L^3}{16}\right) = -\frac{PL^3}{48EI}
\]

And from the right:

\[
 w_{\text{max}}^R(x_1 = L/2) = \frac{P}{2EI} \left(\frac{L^3}{16} + \frac{L^3}{48} - \frac{L^3}{8}\right) = -\frac{PL^3}{48EI}
\]

The deflection profile looks like this:
S9. \( r^2 = x_1^2 + x_2^2 \), so \( I_r = I_{11} + I_{22} \).

S10. Simply substitute Eqn. (34) to Eqns. (39)-(42).

\[
I_{11}' = \int (x_2')^2 dA = \int (x_2 - S_2)^2 dA = \int (x_2^2 - 2x_2S_2 + S_2^2) dA = I_{11} - 2S_2i_1 + S_2^2 A
\]

\[
I_{22}' = \int (x_1')^2 dA = \int (x_1 - S_1)^2 dA = \int (x_1^2 - 2x_1S_1 + S_1^2) dA = I_{22} - 2S_1i_2 + S_1^2 A
\]

\[
I_{12}' = \int x_1'x_2' dA = \int (x_1 - S_1)(x_2 - S_2) dA = \int (x_1x_2 + S_1S_2 - x_1S_2 - x_2S_1) dA = I_{12} + S_1S_2 A - S_2i_2 - S_1i_1
\]

\[
I_r' = I_{11}' + I_{22}' = I_{11} + I_{22} - 2(S_2i_1 + S_1i_2) + (S_1^2 + S_2^2) A
\]

If the new CS has centroid at the origin, then one can use Eqns. (37) and (38) to obtain:

\[
i_1 = S_2 A
\]

\[
i_2 = S_1 A
\]

so that

\[
I_{11}' = I_{11} - S_2^2 A \quad (144)
\]

\[
I_{22}' = I_{22} - S_1^2 A \quad (145)
\]

\[
I_{12}' = I_{12} - S_1S_2 A \quad (146)
\]

\[
I_r' = I_{11} + I_{22} - (S_1 + S_2) A \quad (147)
\]

S11. Refer to the diagram below.

![Diagram](image)

The centroid is at the intersection of the two lines of symmetry. Taking the centroid as the origin:

\[
I_{11} = \int x_2^2 dA
\]

It is natural to take \( dA = Wdx_2 \), so that

\[
I_{11} = \int_{-H/2}^{H/2} x_2^2 W dx_2 = W \frac{1}{3} \left[ \frac{H^3}{2} + \frac{1}{8} \right] = \frac{WH^3}{12}
\]

From symmetry

\[
I_{22} = \frac{HW^3}{12}
\]

It is clear that from symmetry \( I_{12} = 0 \). But one can check it as
\[ I_{12} = \int_{-H/2}^{H/2} \int_{W/2}^{-W/2} x_1 x_2 \, dx_1 \, dx_2 = \int_{-H/2}^{H/2} \frac{x_1^2}{2} \bigg|_{-W/2}^{W/2} \, x_2 \, dx_2 = \int_{-H/2}^{H/2} \frac{W^2}{2} \left( 1 - \frac{1}{4} \right) x_2 \, dx_2 = 0 \]

**S12.** Refer to the diagram below.

First need to find the centroid.

\[ i_1 = \int x_2 \, dA \]

It is easiest to represent \( dA \) as shown in the diagram \( dA = adx_2 \), where \( a \) is the length of the hatched strip.

From triangles

\[ \frac{W}{H} = \frac{a}{H - x_2} \]

or

\[ a = \frac{W}{H} (H - x_2) \]

So that

\[ i_1 = \int_0^H x_2 \, \frac{W}{H} (H - x_2) \, dx_2 = \frac{W}{H} \int_0^H \left( x_2 H - x_2^2 \right) \, dx_2 = \frac{W}{H} \left( \frac{H^2}{2} H - \frac{H^3}{3} \right) = \frac{WH^2}{6} \]

From symmetry

\[ i_2 = \frac{HW^2}{6} \]

Hence the centroid is at

\[ S_1 = \frac{i_2}{A} = \frac{HW^2/2}{6HW} = \frac{W}{3} \]

Again from symmetry

\[ S_2 = \frac{H}{3} \]

When calculating the second moments one has a choice: (a) calculating wrt to the centroid directly, or (b) calculating wrt the original CS, and then translating to the centroid using solution to ex. prob. 10.

First, (a) calculating directly. Because the origin has shifted one now has
The inner integral is
\[ \frac{a}{2/3H - x_2} \]

or
\[ a = \frac{W}{H} \left( \frac{2}{3} H - x_2 \right) \]

So
\[
I_{11} = \int_{-H/3}^{2/3H} x_1^3 \left( \frac{W}{H} \left( \frac{2}{3} H - x_2 \right) \right) dx_2 = \frac{W}{H} \int_{-H/3}^{2/3H} \left( \frac{2}{3} H x_2^2 - x_3^3 \right) dx_2 = \frac{W}{H} \left[ \frac{2}{3} H \left( \frac{x_2^3}{3} - \frac{x_3^4}{4} \right) \right]_{-H/3}^{2/3H}
\]

\[
= \frac{W}{H} \left[ \frac{2}{9} \left( \frac{8}{27} H^3 + \frac{1}{27} H^3 \right) - \frac{1}{4} \left( \frac{16}{81} H^4 - \frac{1}{81} H^4 \right) \right] = WH^3 \left[ \frac{2}{27} - \frac{15}{4 \cdot 81} \right] = \frac{WH^3}{36}
\]

From symmetry
\[ I_{22} = \frac{WH^3}{36} \]

Now, (b) calculating \( I_{11} \) in the original CS:
\[
I_{11} = \int_{0}^{H} x_2^3 \left( \frac{W}{H} \left( H - x_2 \right) \right) dx_2 = \frac{W}{H} \int_{0}^{H} \left( H x_2^2 - x_3^3 \right) dx_2 = \frac{W}{H} \left( \frac{H^4}{3} - \frac{H^4}{4} \right) = \frac{WH^3}{12}
\]

and then shifting it to the centroid using expressions from ex. prob. S10:
\[
I_{11}' = I_{11} - S_2 A = \frac{WH^3}{12} - \left( \frac{H^2}{3} \right) \frac{WH^2}{2} = WH^3 \left( \frac{1}{12} - \frac{1}{18} \right) = \frac{WH^3}{36}
\]

which matches the answer found by method (a). The same can be obtained for \( I_{22} \).

Finally, calculating \( I_{12} \) directly. The most important point is to choose the limits of integration right. Refer to the diagram.

\[
I_{12} = \int_{x_2=-H/3}^{x_2=2/3H} \int_{x_1=-W/3}^{x_1=W/3} x_1 x_2 dx_1 dx_2
\]

Using the expression for \( a \) above
\[
a - \frac{W}{3} = \frac{2}{3} W - \frac{1}{3} W - \frac{W}{H} x_2 = \frac{W}{3} - \frac{W}{H} x_2
\]

Because the upper integration limit for \( x_1 \) depends on \( x_2 \), the integration must be done in the correct order
\[
I_{12} = \int_{x_2=-H/3}^{x_2=2/3H} \left( \int_{x_1=-W/3}^{x_1=W/3} x_1 x_2 dx_1 \right) x_2 dx_2
\]

The inner integral is
\[
\int_{-W/3}^{W/3} \left( \int_{x_2=-W/3}^{x_2=2/3H} x_2^2 dx_2 \right) x_2 dx_2 = \frac{W^2}{2H^2} \left( \frac{x_2^4}{4} - \frac{x_2^3}{3} \right)_{-W/3}^{2/3H} = \frac{W^2 H^2}{2} \left[ \frac{1}{4} \left( \frac{16}{81} - \frac{1}{81} \right) - \frac{1}{9} \left( \frac{8}{27} + \frac{1}{27} \right) \right]
\]

So that
\[
I_{12} = \frac{W^2 H^2}{2} \left( \frac{15}{4 \cdot 81} - \frac{2}{27} \right) = \frac{W^2 H^2}{2} \cdot \frac{15 - 2 \cdot 4 \cdot 3}{8 \cdot 81} = \frac{W^2 H^2}{2} \cdot \frac{15 - 24}{8 \cdot 81} = -\frac{W^2 H^2}{72}
\]

Note that \( I_{12} \) is negative!
And now, calculating $I_{12}$ in the original CS and then shifting it to the centroid. Note how the integration limits change.

$$I_{12} = \int_0^H \int_0^{\frac{W}{H} (H-x_2)} x_1 x_2 dx_1 dx_2 = \int_0^H \left( \int_0^{\frac{W}{H} (H-x_2)} x_1 dx_1 \right) x_2 dx_2$$

The inner integral is

$$\left. \frac{x_1^2}{2} \right|_0^{\frac{W}{H} (H-x_2)} = \frac{W^2}{2H^2} \left( H^2 - 2Hx_2 + x_2^2 \right)$$

So that the outer integral becomes

$$I_{12} = \frac{W^2}{2H^2} \int_0^H \left( H^2 x_2 - 2Hx_2^2 + x_2^3 \right) dx_2 = \frac{W^2}{2} \left( \frac{1}{2} - 2 \frac{1}{3} + \frac{1}{4} \right) = \frac{W^2}{2} \frac{6 - 8 + 3}{12} = \frac{W^2H^2}{24}$$

From ex. prob. 10:

$$I_{12}' = I_{12} - S_1S_2 A = \frac{W^2H^2}{24} - \frac{WH}{9} \cdot \frac{WH}{2} = \frac{W^2}{2} \left( \frac{1}{24} - \frac{1}{18} \right) = \frac{W^2}{2} \frac{3 - 4}{72} = -\frac{W^2H^2}{72}$$

which matches the answer found by the direct method.

**S13.** Make sure to solve ex. probs. 11 and 12 before attempting this.

The area, and hence the mass per unit length, of the triangular cross section is $1/2$ of that of the rectangular. However, $I_{11}$ and $I_{22}$ for a triangle are 3 times smaller. In addition $x^{\text{max}}$ in a rectangle is $H/2$, but $2/3H$ in a triangle. So the maximum (or minimum) axial stress in a beam with a triangular cross section is $3 \times \frac{2}{1/2} = 4$ times higher than in a beam of a rectangular cross section. An engineer will have to make a judgment on whether the saving in mass is justified.

In addition, note that the triangular cross section has no axis of symmetry, which is reflected by the fact that $I_{12} \neq 0$.

**S14.** Due to axisymmetry, the centroid is at the centre of the circle. First calculate the polar second moment.

$$I_r = \int r^2 dA$$

$$dA = 2\pi rdr$$

so that

$$I_r = \int_0^R 2\pi r^3 dr = \frac{\pi R^4}{2}$$
From ex. prob. 9:

\[ I_{11} = I_{22} = \frac{1}{2} I_r = \frac{\pi R^4}{4} \]

Due to axisymmetry \( I_{12} = 0 \) in any CS with the origin at centroid.

**S15.** Use the expressions for the circle, ex. prob. 14.

![Diagram of a ring and circle](image)

The outer radius of the ring is \( R_o \) and \( R_i \) is the inner radius.

Since the area integrals are additive, use superposition:

\[ I_r = I_{\text{ring}}(R_o) - I_{\text{ring}}(R_i) = \frac{\pi}{2} (R_o^4 - R_i^4) \]

and

\[ I_{11} = I_{22} = \frac{\pi}{4} (R_o^4 - R_i^4) \]

Alternatively can calculate directly as in ex. prob. 14, but with different BC:

\[ I_r = \int_{R_i}^{R_o} 2\pi r^3 dr = 2\pi \left[ \frac{r^4}{4} \right]_{R_i}^{R_o} = \frac{\pi}{2} (R_o^4 - R_i^4) \]

**S16.** Use the expressions for the second moments from ex. probs. 14 and 15. Let

\[ R_i = \alpha R_o \]

where \( 0 \leq \alpha < 1 \), so that at \( \alpha = 0 \) the ring becomes a circle. Let further assume that the outer radii of the ring and of the circle are equal, \( R_o = R \). Then

\[ I_{\text{ring}} \] / \( I_{\text{circ}} = \frac{R^4 - (\alpha R)^4}{R^4} = 1 - \alpha^4 \]

Note that this expression is valid for all second moments, so we don’t need to specify the subscripts.

Ratio of mass per unit length for beams of both cross sections is:

\[ \frac{m^\text{ring}}{m^\text{circ}} = \frac{A^\text{ring}}{A^\text{circ}} = \frac{R^2 - (\alpha R)^2}{R^2} = 1 - \alpha^2 \]

Because the outer radii are equal in both cases, only change in \( I \) needs to be considered when calculating \( t^\text{max} \). From Eqn. (28):

\[ \frac{t^\text{ring}}{t^\text{circ}} = \frac{I^\text{circ}}{I^\text{ring}} = \frac{1}{1 - \alpha^4} \]

One can plot these ratios against \( \alpha \) to get a complete picture. Here we examine only two values of \( \alpha \).
So for $\alpha = 0.9$, i.e. pipe wall thickness of 0.1 of the outer radius, the beam will be over five times lighter. The price of this saving in mass is that the maximum stress will be almost three times higher. The potential for optimisation is clear.

S17. Refer to the diagram:

1. solid circle. $A = \pi r^2 \rightarrow r = \sqrt{A/\pi}$.

$$I_{11} = \frac{\pi A^2}{4 \pi^2} = \frac{1}{4 \pi} A^2$$

$$t_{\text{max}} = \frac{Mr}{I_{11}} = \frac{4\pi\sqrt{A}}{A^2\sqrt{\pi}} M = 4\sqrt{\pi}MA^{-3/2}$$

$$t_{\text{max}} = 7.090 \times MA^{-3/2}$$

2. solid square. width = height = $\sqrt{A}$. $I_{11} = \frac{A^2}{12}$.

$$t_{\text{max}} = \frac{M\sqrt{A}/2}{I_{11}} = 6 \times MA^{-3/2}$$

3. ring. Ring area:

$$A = \pi (R_o^2 - R_i^2)$$

Assume $R_o = W/2 \rightarrow R_o^2 = A$. Hence $R_i^2 = A - A/\pi$.

$$I_{11} = \frac{\pi}{4} \left( A^2 - A^2 (1 - 1/\pi)^2 \right) = \frac{\pi A^2}{4} \left( \frac{2}{\pi} - \frac{1}{\pi^2} \right) = \frac{2\pi - 1}{4\pi} A^2$$

$$t_{\text{max}} = \frac{MR_o}{I_{11}} = \frac{4\pi\sqrt{A}}{(2\pi - 1)A^2} M = \frac{4\pi}{2\pi - 1} MA^{-3/2} = 2.379 \times MA^{-3/2}$$

4. square box. Assume the box shape as this:
So $Y^2 = W^2 - A = 3A$.

$$I_{11} = \frac{1}{12} (W^4 - Y^4) = \frac{1}{12} (16A^2 - 9A^2) = \frac{7}{12} A^2$$

$$t_{\text{max}} = \frac{MW/2}{I_{11}} = \frac{12\sqrt{A}}{7A^2} M = \frac{12}{7} MA^{-3/2} = 1.714 \times MA^{-3/2}$$

5. I-section.

![I-section diagram]

The area is approx. $A = 3Wt$, where $t$ is thickness of the web. So $t = \sqrt{A/6}$.

For this section

$$I_{11} = I_{\text{square}} - 2I_{\text{rect}} = \frac{W^4}{12} - 2 \frac{((W - t)/2)(W - 2t)^3}{12} = \frac{1}{12} \left[ 16A^2 - (2\sqrt{A} - \sqrt{A/6})(2\sqrt{A} - 2\sqrt{A/6})^3 \right]$$

$$= \frac{A^2}{12} \left[ 16 - (2 - \frac{1}{6})(2 - \frac{1}{3})^3 \right] = \frac{A^2}{12} \left[ 16 - \frac{11}{6} \cdot \frac{5}{3} \right] = 0.626A^2$$

$$t_{\text{max}} = \frac{MW/2}{I_{11}} = 1.597 \times MA^{-3/2}$$

So we can achieve over 4 times reduction in stress by choosing the cross section wisely.

Note that, in contrast to the first four sections, I-section works well in only one orientation. If I-section is fitted by mistake in the opposite orientation, then:

$$I_{22} = \frac{2tW^3}{12} + \frac{(W - 2t)h^3}{12} = A^2 \left( \frac{8}{6} \cdot \frac{2 - 1/3}{12 \cdot 6} \right) = 0.223A^2$$

$$t_{\text{max}} = \frac{MW/2}{I_{22}} = 4.484 \times MA^{-3/2}$$

So the maximum stresses in this orientation will be nearly 3 times greater than in the best orientation. Or, in other words, the I-section oriented badly is still better than the solid square, but worse than ring.

Other popular cross sections used in construction:

- S18. First I need to find the centroid of the cross section. I use a CS with the origin at the outer corner of the cross section. I split the section into two parts, A and B, and find centroids and the second moments of each part wrt the chosen CS. I use expressions from ex. prob. 10.
In that CS coordinates of the centroids of A and B are easily found, e.g. along axis 2: \( S^A_2 = \frac{W}{2}, S^B_2 = \frac{T}{2} \).

The total area is \( A = WT + (W - T)T = T(2W - T) \).

If \( W = 40 \text{ mm} \) and \( T = 4 \text{ mm} \), then \( A = 304 \text{ mm}^2 = 3 \text{ cm}^2 \).

From (46) the coordinate of the centroid of the whole cross section along axis 2 is:

\[
S_2 = \frac{1}{A} (S^A_2 A^A + S^B_2 A^B) = \frac{W}{2} \frac{WT + \frac{T}{2} (W - T)T}{T(2W - T)} = \frac{W^2 + T(W - T)}{2(2W - T)}
\]

The second moments of areas A and B wrt their centroids are:

\[
I^A_{11} = \frac{TW^3}{12}, \quad I^B_{11} = \frac{(W - T)T^3}{12}
\]

We use (144) to calculate the second moments of these areas wrt the chosen CS. Note that in (144) the primed CS is the one that passes through the centroid, and the unprimed CS is the starting CS. In our case we need to do inverse calculation - we know the second moment in the CS that passes through the centroid, but want to find out the second moments in the chosen (global) CS. So we need to original CS is:

\[
I^A_{11} = I^A_{11} \text{(centroid)} + (S^A_2)^2 A^A = \frac{TW^3}{12} + \frac{W^2}{4} WT = \frac{TW^3}{3}
\]

\[
I^B_{11} = I^B_{11} \text{(centroid)} + (S^B_2)^2 A^B = \frac{(W - T)T^3}{12} + \frac{T^2}{4} (W - T)T = \frac{(W - T)T^3}{3}
\]

The second moment for the whole cross section, in the chosen (global) CS, about axis 1 is

\[
I_{11} = I^A_{11} + I^B_{11} = \frac{T}{3} (W^3 + (W - T)T^2)
\]

Now we can find the second moment of the whole cross section wrt to the centroid of the whole cross section, about axis 1. From (144):
\[ I_{11}' = I_{11} - S^2 A = \frac{T}{3} (W^3 + (W - T)T^2) - \left(\frac{W^2 + T(W - T)}{2(2W - T)}\right)^2 T(2W - T) \]

\[ = \frac{T}{3} \left[ W^3 + (W - T)T^2 - 3 \left(\frac{W^2 + T(W - T)}{4(2W - T)}\right)^2 \right] \]

For our values of \( W \) and \( T \): \( I_{11} = 4.6 \times 10^4 \text{ mm}^4 \) or 4.6 cm\(^4\).

From symmetry, \( I_{22} = I_{11} \).

For \( I_{12} \) it is easier to calculate first \( I_{A12}' \) and \( I_{B12}' \) and then calculate \( I_{12}' = I_{A12}' + I_{B12}' \). This is because for rectangles \( I_{12} = 0 \) when 1 and 2 are symmetry axes passing through centroid. Note that we do not use the global (unprimed) CS for \( I_{12} \) at all. From (146):

\[ I_{A12}' = -S_{C-C_A} S_{C-C_A} A^A \]
\[ I_{B12}' = -S_{C-C_B} S_{C-C_B} A^B \]

where superscripts \( C - C_A \) and \( C - C_B \) and refer to the distances between the centroids of \( A \) and \( B \) respectively, and the centroid of the whole cross section, \( C \).

\[ S_{C-C_A} = \frac{W^2 + T(W - T)}{2(2W - T)} - \frac{T}{2} \]
\[ S_{C-C_B} = \frac{W^2 + T(W - T)}{2(2W - T)} - \frac{W}{2} \]

For the values above, \( I_{A12}' = -1.3 \times 10^4 \text{ mm}^4 \).

\[ S_{C-C_A} = \frac{W^2 + T(W - T)}{2(2W - T)} - \left(\frac{T + W - T}{2}\right) \]
\[ S_{C-C_B} = \frac{W^2 + T(W - T)}{2(2W - T)} - \frac{T}{2} \]

For the values above, \( I_{B12}' = -1.4 \times 10^4 \text{ mm}^4 \).

Finally \( I_{12}' = -2.7 \times 10^4 \text{ mm}^4 \).

The following makes sense only after completing Sec. 6.2.

The principal values and directions from Lapack DSYEV:

Original tensor
\[
\begin{bmatrix}
0.00000000000E+00 & 0.00000000000E+00 & 0.00000000000E+00 \\
0.00000000000E+00 & 4.60000000000E+00 & 2.70000000000E+00 \\
0.00000000000E+00 & 2.70000000000E+00 & 4.60000000000E+00
\end{bmatrix}
\]

The DSYEV eigenvalues in increasing order
\[
\begin{bmatrix}
0.00000000000E+00 & 1.90000000000E+00 & 7.30000000000E+00
\end{bmatrix}
\]

The DSYEV orthonormal eigenvectors (columns)
\[
\begin{bmatrix}
1.00000000000E+00 & 0.00000000000E+00 & 0.00000000000E+00 \\
0.00000000000E+00 & -7.07106781187E-01 & 7.07106781187E-01 \\
0.00000000000E+00 & 7.07106781187E-01 & 7.07106781187E-01
\end{bmatrix}
\]

The angles (deg)
\[
\begin{bmatrix}
0.00000000000E+00 & 9.00000000000E+00 & 9.00000000000E+00 \\
9.00000000000E+01 & 1.35000000000E+02 & 4.50000000000E+01 \\
9.00000000000E+01 & 4.50000000000E+01 & 4.50000000000E+01
\end{bmatrix}
\]

So the principal values of the second moments are \( 7.3 \times 10^4 \text{ mm}^4 \) and \( 1.9 \times 10^4 \text{ mm}^4 \) and the angle is \( 45^5 \), as expected for \( I_{11} = I_{22} \).
S19. Use Eqn. (49) and an expression for \( I \) from ex. prob. 15:

\[
M_{\text{max}} = \frac{\sigma_f I_{11}}{x_{\text{max}}^2} = \frac{\sigma_f \pi (R_o^4 - R_i^4)}{4R_o} = \frac{500 \times 3.14 \times (10^4 - 8^4)}{40} = 231,732 \text{ N m} = 232 \text{ N m}
\]

S20. Need to differentiate \( w \) three times:

\[
w' = \frac{dw}{dx_1} = -zC_1 \sin zx_1 + zC_2 \cos zx_1
\]

\[
w'' = \frac{d^2 w}{dx_1^2} = -z^2 C_1 \cos zx_1 - z^2 C_2 \sin zx_1
\]

\[
w''' = \frac{d^3 w}{dx_1^3} = z^3 C_1 \sin zx_1 - z^3 C_2 \cos zx_1
\]

Putting all this into Eqn. (56):

\[
\frac{d^3 w}{dx_1^3} + z^2 \frac{dw}{dx_1} = z^3 C_1 \sin zx_1 - z^3 C_2 \cos zx_1 + z^2 (-zC_1 \sin zx_1 + zC_2 \cos zx_1) = 0
\]

S21. A clamped/clamped buckled column will look like this:

There are five BCs:
1. \( w(x_1 = 0) = 0 \)
2. \( w'(x_1 = 0) = 0 \)
3. \( w'(x_1 = l/2) = 0 \)
4. \( w(x_1 = l) = 0 \)
5. \( w'(x_1 = l) = 0 \)

From Eqn. (57) in Sec. 4.5.1:

\[w = C_1 \cos z l + C_2 \sin z l + C_3\]

Using BC 2:

\[C_2 = 0\]

Using BC 1:

\[C_1 + C_3 = 0\]

Using BC 4:

\[C_1 \cos z l + C_3 = 0\]

From the last two equations:

\[C_1 \cos z l - C_1 = 0\]

\(C_1 = 0\) is a trivial solution corresponding to an unbuckled, straight, column. This solution is of no interest. Hence we must have:

\[\cos z l = 1\]

or
The magnitude of the critical load is:

\[ P_{\text{crit}} = (2n\pi)^2 \frac{EI}{l^2} \]

The lowest critical load is when \( n = 1 \):

\[ P_{\text{crit}}^{\text{lowest}} = 4\pi^2 \frac{EI}{l^2} \]

Note that the lowest critical load is 4 times higher for the clamped/clamped BC compared to the pin/pin BC, Eqn. (58) in Sec. 4.5.1.

\[ S22. \] For a pin/pin column, from Eqn. (59):

\[ w = C \sin \frac{n\pi x_1}{l} \]

For a clamped/clamped column, from ex. prob. 21:

\[ w = C(\cos \frac{2n\pi x_1}{l} - 1) \]

The \( n = 1 \) shapes are drawn to scale, i.e. the maximum values of deflection are \( w/C = 1 \). All other shapes are drawn with smaller magnitude to ease understanding.
S23. From Eqn. (53) in Sec. 4.5.1:

\[ \varepsilon_{11} = x_2 \frac{d^2 w}{dx_1^2} + \frac{P}{EA} \]

or, since \( P < 0 \), we can rewrite this as:

\[ \varepsilon_{11} = x_2 \frac{d^2 w}{dx_1^2} - \frac{|P|}{EA} \]

The stress state is uniaxial, so

\[ t = E \varepsilon_{11} = Ex_2 \frac{d^2 w}{dx_1^2} - \frac{|P|}{A} \]

or using Eqn. (54):

\[ t = \frac{Mx_2}{I} - \frac{|P|}{A} \]

or, since \( M = |P|w \):

\[ t = |P|\left( \frac{wx_2}{I} - \frac{1}{A} \right) \]

Buckling with \( P_{\text{crit}} \) produces neutral equilibrium. In other words any \( w \) value is possible, provided the deflections remain small. Hence we cannot know the exact answer, and only a qualitative answer is possible.

The stress profile consists of two superimposed fields - bending and compression. The magnitude of compression stress is fixed, but the magnitude of the bending stress can vary. Hence we can have a situation where the maximum stress is positive or negative.

S24. Refer to the drawing:
A surface element is defined by its area and its unit normal. One can combine the two into a single vector:

$$S = S n$$

A projection is clearly determined by the relative orientation of the normals of the two planes:

$$S^p = |S \cdot n^p| = |S| |n \cdot n^p|$$

Let’s see if this is correct on two extreme cases: if $n||n^p \rightarrow S^p = S$, which is correct. If $n \perp n^p \rightarrow S^p = 0$, which is also correct. Finally, what if $n^p$ matches a basis vector, i.e. $n^p = e^i$? If $n^p = e^1 = (1, 0, 0)^T$ then $|n \cdot n^p| = n_1$ and $S^p = Sn_1$. Similarly for the two other cases: if $n^p = e^2 = (0, 1, 0)^T$ then $S^p = Sn_2$. If $n^p = e^3 = (0, 0, 1)^T$ then $S^p = Sn_3$.

**S25.** Remember that the stress tensor is symmetric. Remember to use the sign convention, sec. 5.2.

---

**S26.** $\nabla$ is a differential operator, a vector called *gradient*, defined as:

$$\nabla = \frac{\partial}{\partial x_1} e^1 + \frac{\partial}{\partial x_2} e^2 + \frac{\partial}{\partial x_3} e^3$$

where $e^i$ are the basis vectors. Consult your maths lectures and books for more.

So $\nabla$ is a vector operator. $\nabla \cdot$ is a dot (or inner) product, meaning that $\nabla$ is multiplied by another vector or tensor. In case of a vector $\mathbf{a} = a_i$:

$$\nabla \cdot \mathbf{a} = \begin{pmatrix} \frac{\partial}{\partial x_1} , \frac{\partial}{\partial x_2} , \frac{\partial}{\partial x_3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = a_{i, i}$$
So the result is a scalar, or a rank 0 tensor. Hence, $\nabla \cdot$ reduces the rank of the result by one, compared to the rank of the tensor to which it is applied.

For a R2T, $T = T_y$:

$$\nabla \cdot T = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{21}}{\partial x_2} + \frac{\partial T_{31}}{\partial x_3} \\ \frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{32}}{\partial x_3} \\ \frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{23}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \end{pmatrix}$$

or, for a transposed tensor:

$$\nabla \cdot T^T = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \end{pmatrix}$$

Note that from R2T we got rank 1 vector. So

$$\nabla \cdot \sigma = \left\{ \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3}, \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3}, \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \right\}$$

On the other hand, in $\sigma_{i,j} \ j$ is a dummy (summation) index. Only a single index remains, $i$. Hence the result of this spatial derivation is a rank 1 tensor - a vector:

$$\sigma_{i,j} = \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} = \begin{pmatrix} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} \end{pmatrix}$$

Differentiating instead over the first subscript gives:

$$\sigma_{j,i} = \sigma_{1,j,1} + \sigma_{2,j,2} + \sigma_{3,j,3} = \begin{pmatrix} \sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} \\ \sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} \\ \sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} \end{pmatrix}$$

Since $\sigma = \sigma^T$, both derivatives give the same vector.

In vector calculus $\nabla \cdot a$ is called divergence of vector field $a$. The index equivalent of $\nabla \cdot$ operator is $b_{\ldots}\ldots$, where $b_{\ldots}$ is a tensor of arbitrary rank.

S27. The difference is easy to see if one uses the index notation:

$$\nabla \cdot Z = Z_{\ldots,p}$$

whereas

$$\nabla Z = Z_{\ldots\ldots}$$

In the first case $p$ is a dummy index, so the resulting tensor is of rank $r - 1$, where $r$ is the rank of $Z$. In the second case $p$ is a live index, so the rank of the resulting tensor is $r + 1$.

Say $Z$ is a rank 1 tensor, i.e. a vector, $z_i$; Then
\[
\n\nabla \cdot z = z_{i,i} = \frac{\partial z_1}{\partial x_1} + \frac{\partial z_2}{\partial x_2} + \frac{\partial z_3}{\partial x_3}
\]

which is a scalar, whereas

\[
\nabla z = z_{i,j} = \left( \frac{\partial z_1}{\partial x_1} \right)_{i,j} + \left( \frac{\partial z_2}{\partial x_1} \right)_{i,j} + \left( \frac{\partial z_3}{\partial x_1} \right)_{i,j}
\]

which is a R2T.

For a scalar field \( z \), \( \nabla z \) is the gradient of the field, a vector pointing in the direction of the maximum increase of \( z \).

**S28.** Multiply Eqn. (63) by an arbitrary vector \( a_j \):

\[
\sigma_{ij} n_i a_j = t_{n_j} a_j
\]

The right hand side is a projection of one vector to another. This is a scalar and an invariant. Hence the left hand side must also be constant. Since both \( n_i \) and \( a_i \) are arbitrary, then by definition \( \sigma_{ij} \) is a R2T.

**S29.** Operations on tensor of rank 1 and 2 can be done with conventional column and row vectors and square matrices. So

\[
\begin{align*}
b_i &= B_{ij} a_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = a_i \\
c_i &= C_{ij} a_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\end{align*}
\]

So \( |c| = |a| \), hence \( C_{ij} \) is an example of a pure rotation tensor, one that changes the orientation of a vector, but leaves its length intact.

\[
\begin{align*}
d_i &= D_{ij} a_j = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}
\end{align*}
\]

So that \( d = 2a \). Thus \( D_{ij} \) is an example of a stretch tensor, one that changes the length of a vector, but not its orientation.

**S30.** Let’s assume that we rotate the CS clockwise:
The cosines of angles between the new and the old basis vectors are then:

\[
\cos \angle(e'_1, e^1) = \cos \beta; \quad \cos \angle(e'_1, e^2) = \cos(\pi/2 + \beta) = -\sin \beta; \quad \cos \angle(e'_1, e^3) = 0
\]

\[
\cos \angle(e'_2, e^1) = \cos(\pi/2 - \beta) = \sin \beta; \quad \cos \angle(e'_2, e^2) = \cos \beta; \quad \cos \angle(e'_2, e^3) = 0
\]

\[
\cos \angle(e'_3, e^1) = 0; \quad \cos \angle(e'_3, e^2) = 0; \quad \cos \angle(e'_3, e^3) = 1
\]

So

\[
R = \begin{pmatrix}
\cos \beta & -\sin \beta & 0 \\
\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

We must have \(a' = R \cdot a\). Let’s check:

\[
R \cdot a = \begin{pmatrix}
\cos \beta & -\sin \beta & 0 \\
\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
0
\end{pmatrix} = \begin{pmatrix}
a_1 \cos \beta - a_2 \sin \beta \\
a_1 \sin \beta + a_2 \cos \beta \\
0
\end{pmatrix}
\]

Looking at the diagram above, this is correct.

Remember the form of \(R\) for a simple rotation about only one coordinate axis.

S31. Remember that the basis vectors are orthogonal in orthonormal CS:

\[e^i \cdot e^j = I = \delta_{ij}\]

In a rotated CS:

\[e^{i'} \cdot e^{j'} = R_{ip}e^p \cdot R_{jq}e^q = R_{ip}R_{jq}e^p \cdot e^q = R_{ip}R_{jq}\delta_{pq} = R_{ip}R_{jp} = R_{ip}R_{Tpj} = \delta_{ij}\]

So

\[R \cdot R^T = I = R \cdot R^{-1} \iff R^T = R^{-1}\]

S32. By definition:

\[a' = R \cdot a\]

Multiply both sides by \(R^T\) from the left:

\[R^T \cdot a' = R^T \cdot R \cdot a\]

From ex. prob. 31: \(R^T \cdot R = I\), hence:
\[ \mathbf{R}^T \cdot \mathbf{a}' = \mathbf{a} \]

In index notation the same proof looks like this:

\[ a'_i = R_{ij} a_j \]

Multiply both sides by \( R^T_{pi} \) from the left:

\[ R^T_{pi} a'_i = R^T_{pi} R_{ij} a_j \]

From ex. prob. 31: \( R^T_{pi} R_{ij} = \delta_{pj} \), hence:

\[ R^T_{pi} a'_i = \delta_{pj} a_j = a_p \]

**S33.** Start from Eqn. (75). If \( \mathbf{T} \) is \( \mathbf{R}^T \) then \( \forall a_i, b_i \):

\[ T_{ij} a_i b_j \equiv \text{const} \]

That means that this product will be same in any other CS:

\[ T'_{ij} a'_i b'_j = T_{ij} a_i b_j \]

From ex. prob. 32: \( a_i = R^T_{pi} a'_p, b_j = R^T_{pj} b'_q \), so that

\[ T'_{ij} a'_i b'_j = T_{ij} R^T_{pi} a'_p R^T_{pj} b'_q \]

Both indices on the left are dummy, and hence can be changed at will. I want to change \( i \rightarrow p, j \rightarrow q \):

\[ T'_{pq} a'_p b'_q = T_{ij} R^T_{pi} a'_p R^T_{pj} b'_q \]

Provided that \( a'_p \neq 0 \) and \( b'_q \neq 0 \), we can divide by these vectors:

\[ T'_{pq} = T_{ij} R^T_{pi} R^T_{jq} = R_{pi} R_{jq} T_{ij} \]

In tensor notation the same proof looks like this:

\[ (\mathbf{T} \cdot \mathbf{a}) \cdot \mathbf{b} \equiv \text{const} \]

Explain why the brackets are needed in the above expression.

\[ (\mathbf{T}' \cdot \mathbf{a}') \cdot \mathbf{b}' = (\mathbf{T} \cdot \mathbf{a}) \cdot \mathbf{b} \]

From ex. prob. 32: \( \mathbf{a} = \mathbf{R}^T \cdot \mathbf{a}', \mathbf{b} = \mathbf{R}^T \cdot \mathbf{b}' \), so

\[ (\mathbf{T}' \cdot \mathbf{a}') \cdot \mathbf{b}' = (\mathbf{T} \cdot \mathbf{R}^T \cdot \mathbf{a}') \cdot \mathbf{R}^T \cdot \mathbf{b}' \]

Provided that \( \mathbf{b}' \neq 0 \), we can divide by it:

\[ \mathbf{T}' \cdot \mathbf{a}' = (\mathbf{T} \cdot \mathbf{R}^T \cdot \mathbf{a}') \cdot \mathbf{R}^T \]

Then think of tensors as of matrices, i.e. the order matters. Multiply both sides by \( \mathbf{R} \) from the right:

\[ \mathbf{T}' \cdot \mathbf{a}' \cdot \mathbf{R} = (\mathbf{T} \cdot \mathbf{R}^T \cdot \mathbf{a}') \cdot \mathbf{R}^T \cdot \mathbf{R} \]

Remember that \( \mathbf{R} \) is orthogonal, so that \( \mathbf{R}^T \cdot \mathbf{R} \) on the right disappears:

\[ \mathbf{T}' \cdot \mathbf{a}' \cdot \mathbf{R} = \mathbf{T} \cdot \mathbf{R}^T \cdot \mathbf{a}' \]

Both sides are vectors. Transpose of a vector is the same vector; \( \mathbf{a}^T = \mathbf{a} \). Hence we can transpose just one side of the equation, not both. Let’s transpose the right hand side:

\[ \mathbf{T}' \cdot \mathbf{a}' \cdot \mathbf{R} = \mathbf{a}^T \cdot \mathbf{R} \cdot \mathbf{T}^T \]

Now multiply both sides by \( \mathbf{R}^T \) from the right:

\[ \mathbf{T}' \cdot \mathbf{a}' = \mathbf{a}^T \cdot \mathbf{R} \cdot \mathbf{T}^T \cdot \mathbf{R}^T \]

Finally transpose the right hand side again:

\[ \mathbf{T}' \cdot \mathbf{a}' = \mathbf{R} \cdot \mathbf{T} \cdot \mathbf{R}^T \cdot \mathbf{a}' \]
divide by $a'$ and that’s it.

**S34.** Just think about the meaning of the subscripts, e.g. $M_{ijk}$ is a tensor of rank 3, each component of which has a unique combination of subscripts. Each subscript can take values between 1 and the dimensionality of space, i.e. in 2D space only 1 or 2, and in 3D space 1, 2 or 3. The number of possible combinations is number of possible states (dimensionality of space, $N$) to the power which is the number of indices (rank, $R$). So the number of components is $N^R$. So in 2D space a vector has $2^1 = 2$ components, a rank 2 tensor has $2^2 = 4$ components, a rank 3 tensor has $2^3 = 8$ components and, a rank 4 tensor has $2^4 = 16$ components. In 3D space a vector has $3^1 = 3$ components, a rank 2 tensor has $3^2 = 9$ components, a rank 3 tensor has $3^3 = 27$ components and, a rank 4 tensor has $3^4 = 81$ components.

**S35.** Let’s split $R^2T A$ like this:

$$A = \frac{1}{2} A + \frac{1}{2} A^T - \frac{1}{2} A^T$$

Now, on the right hand side, put together terms 1 and 3, 2 and 4:

$$A = \frac{1}{2} A + \frac{1}{2} A^T + \frac{1}{2} A - \frac{1}{2} A^T$$

or

$$A = \frac{1}{2} (A + A^T) + \frac{1}{2} (A - A^T)$$

The first term is always symmetric, and the second is always anti-symmetric. This might be easier to see in the index notation:

$$(A_{ij} + A_{ji})^T = A_{ji} + A_{ij} = A_{ij} + A_{ji}$$

and

$$(A_{ij} - A_{ji})^T = A_{ji} - A_{ij} = -(A_{ij} - A_{ji})$$

Note that for anti-symmetric tensor all diagonal components are zero: $A_{11} = A_{22} = A_{33} = 0$.

By definition,

$$\text{sym}(A) = \frac{1}{2} (A + A^T)$$

$$\text{asym}(A) = \frac{1}{2} (A - A^T)$$

**S36.** Use the expressions from ex. prob. 35.

$$A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

$$\text{sym}(A) = \frac{1}{2} (A + A^T) = \frac{1}{2} \begin{pmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix}$$

$$\text{asym}(A) = \frac{1}{2} (A - A^T) = \frac{1}{2} \begin{pmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

Let’s check:

$$\text{sym}(A) + \text{asym}(A) = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix} + \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = A$$
**S37.** First:

\[ x_{k,p} = \frac{\partial x_k}{\partial x_p} \]

which in 3D space, i.e. \( k, p = 1, 2, 3 \), is:

\[
x_{k,p} = \begin{pmatrix}
\frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \frac{\partial x_1}{\partial x_3} \\
\frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \frac{\partial x_2}{\partial x_3} \\
\frac{\partial x_3}{\partial x_1} & \frac{\partial x_3}{\partial x_2} & \frac{\partial x_3}{\partial x_3}
\end{pmatrix}
\]

\( x_i = \vec{x} = x \) is a vector giving the coordinate of a point in 3D space. Critically, all 3 coordinates can be chosen independently, i.e. \( x_1, x_2, x_3 \) are 3 independent functions. Since they are independent, by definition, a derivative of one function over another is zero, and a derivative of any function over itself is 1. Hence all diagonal components are 1, and all off-diagonal components are 0:

\[
x_{k,p} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = I = \delta_{kp}
\]

Hence \( \sigma_{jp} x_{k,p} = \sigma_{jp} \delta_{kp} = \sigma_{jk} \).

**S38.** For each \( i \) there are two non-zero terms in this expression: \( ijk \) and \( ikj \), always with opposite signs, so they always cancel each other giving 0. For example, for \( i = 1 \), we have:

\[
e_{1,ji} v_j v_k = e_{123} v_2 v_3 + e_{132} v_3 v_2
\]

all other terms are 0, due to repeated indices.

\[ = v_2 v_3 - v_3 v_2 = 0 \]

**S39.** For each \( i \) this equation means that \( \sigma_{jk} - \sigma_{kj} = 0 \), which means \( \sigma_{jk} = \sigma_{kj} \). For example, for \( i = 3 \):

\[
e_{3,ji} \sigma_{jk} = e_{312} \sigma_{12} + e_{321} \sigma_{21} = \sigma_{12} - \sigma_{21} = 0 \leftrightarrow \sigma_{12} = \sigma_{21}
\]

**S40.** First make a good drawing:

![Diagram](image)

We only need one axis in this example, along the direction of the gravitational pull. Let’s call it 1. Next we need to draw all forces acting on an element of volume \( dV \). Remember that we are in continuum mechanics, where stress is a field, and hence we cannot just draw a force due to gravity in the centre of mass of the body. Instead, consider an arbitrary point \( P \), and volume \( dV \) centred on this point. There is only one force acting on this element: \( dmg = \rho dV g \). This force acts along 1.

The exact placement of axes 2 and 3 is not important. The important fact is that since those are normal to 1, there are no forces acting in 23 plane. Hence immediately we can conclude that

\[ \sigma_{22} = \sigma_{33} = \sigma_{23} = 0 \]
Next we note that the force field is uniform, meaning exactly the same force is acting on any element \(dV\). Hence there can be no shear on other planes either:

\[
\sigma_{12} = \sigma_{13} = 0
\]

Finally, using Eqn (73) we see that only non-zero terms left in the equilibrium equation, Eqn. (66), are:

\[
\sigma_{11,1} = -b_1 + \rho \ddot{x}_1
\]

where \(b_1\) is the body force per unit volume: \(b_1 = \rho g\). For a free fall \(\ddot{x}_1 = g\). Hence these two terms cancel each other:

\[
\sigma_{11,1} = 0
\]

or

\[
\sigma_{11} = \sigma_{11}(x_2, x_3)
\]

And since the body force is uniform, we conclude that \(\sigma_{11}\) is also uniform, i.e. does not depend on coordinate:

\[
\sigma_{11} = \text{const}
\]

To find this integration constant we use the boundary conditions, Eqn. (63):

\[
\sigma_{ij} n_i = t^n_j
\]

We apply this condition to point \(A\) on the boundary, where the normal \(n = (1, 0, 0)\). The traction everywhere on the boundary is zero: \(t^n_j = 0\), hence:

\[
\sigma_{11} n_1 = \sigma_{11} = 0
\]

Therefore \(\sigma = 0\) everywhere in the body. Since the stress state is the same in all points, we call it a uniform stress field.

This might seem like a trivial exercise. However, this is the first solid mechanics problem that we have just solved using the full formalism of the theory, including (1) the equilibrium equations, (2) the boundary conditions and (3) the fact that the stress tensor is symmetric.

**S41.** First step - make a free body diagram, meaning replace all bodies of no interest by their interactions on the body of interest. In this example we replace the ground by the distributed reaction force \(mg/A\), where \(m\) is the mass of the column, \(A\) is its cross section, and \(g\) is the gravitational acceleration.

Tip: always draw a CS:

![Free Body Diagram](image)

Given that there are no forces in 2 and 3 directions, we conclude that only non-zero stress is \(\sigma_{11}\). Such stress states are called uniaxial.

The only remaining equilibrium equation is:

\[
\sigma_{11,1} = -b_1 + \rho \ddot{x}_1
\]
Given that this problem is static, the acceleration, $\ddot{x} = 0$, hence:

$$\sigma_{11,1} = -b_1$$

The body force is solely due to gravity. The body force acting on an element of volume $dV$ is $dmg = \rho dV g$. So the body force per unit volume is $b_1 = \rho g$.

By integrating one obtains:

$$\sigma_{11} = -\rho gx_1 + C$$

The integration constant is found from the traction boundary condition, Eqn. (63):

$$\sigma_{ij} n_i = t^n_j$$

On the top surface of the column $n_i = (-1, 0, 0)$ and $t^n_j = 0$, i.e. no traction. Hence

$$\sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = -\sigma_{11} = 0$$

So

$$\sigma_{11}(x_1 = 0) = 0 \rightarrow C = 0$$

Finally

$$\sigma = 0 \mbox{ except } \sigma_{11} = -\rho gx_1$$

The stress field is uniaxial because there is only a single normal stress component in the body.

The stress field is non-uniform because it changes from point to point.

The minimum stress is at the bottom of the column. If the length of the column is $L$, then this stress is $\sigma_{11}^{\min} = -\rho gL$. Clearly this must match the boundary condition at the bottom. $m = \rho LA$ hence $mg/A = \rho gL$, which is correct.

Note that we have used our sign convention here. The stresses at the bottom face point in the opposite direction to the normal, and hence are negative.

**S42.** As in the previous examples, need to make a good free body diagram:
\[ \sigma_{11,1} = \frac{-F}{LA} \]

or by integrating:

\[ \sigma_{11} = \frac{-F}{LA} x_1 + C \]

As before we find the integration constant from the traction boundary condition, Eqn. (63):

\[ \sigma_{ij} n_i = t^n_j \]

On the top surface of the rocket \( n_i = (-1, 0, 0) \) and \( t^n_j = 0 \), i.e. no traction. Hence

\[ \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = -\sigma_{11} = 0 \]

So

\[ \sigma_{11}(x_1 = 0) = 0 \rightarrow C = 0 \]

Finally

\[ \sigma = 0 \quad \text{except} \quad \sigma_{11} = -\frac{F}{LA} x_1 \]

The stress field is \textit{uniaxial} because there is only a single normal stress component in the body.

The stress field is \textit{non-uniform} because it changes from point to point.

At the bottom end of the rocket, the stresses must match the boundary conditions, which is easy to check by putting \( x_1 = L \) in the above equation.

\textbf{S43.} First step - make a free body diagram. In this example we are only interested in the block on the table. We remove the flat surface and replace it by a vertical reaction force per unit area \( r \). We remove the rope and replace it by a horizontal force per unit area \( mg/HT \). There is also a body force \( b \), due to gravity. Note that we now have only stresses applied to the body.

We need 2 coordinate axes in this example.

![Free body diagram](image)

Note that we have made an assumption that the force from the rope is distributed equally across the whole of the cross section of the block. This assumption is, of course, incorrect, and in practice there will be a complex stress distribution around the point where the rope is attached to the block.

As in the previous example, we note that there are no forces acting along 3, through the plane of the drawing, hence

\[ \sigma_{33} = 0 \]

Also, since we assume a 2D problem, the shear stresses must be zero two:

\[ \sigma_{31} = \sigma_{32} = 0 \]

Let’s write the two remaining equilibrium equations, Eqn. (66), explicitly:

\[ \sigma_{11,1} + \sigma_{12,2} = -b_1 + \rho \ddot{x}_1 \]

\[ \sigma_{21,1} + \sigma_{22,2} = -b_2 + \rho \ddot{x}_2 \]

Let’s assume that the shape of the block remains rectangular. That means there’s no shear deformation, and hence
\( \sigma_{12} = \sigma_{21} = 0 \)

Also, note that along 1 there is no body force, and along 2 there is no acceleration:

\( \sigma_{11,1} = \rho \ddot{x}_1 \)
\( \sigma_{22,2} = -b_2 \)

These equations are now independent and can be solved separately.

To solve along 1, we proceed as in the previous example and find the acceleration from the dynamics of rigid bodies:

\[ \ddot{x}_1 = \frac{F}{m_{\text{block}}} = \frac{mg}{\rho V} = \frac{mg}{\rho HTL} \]

Every term is constant, so by integrating we get:

\[ \sigma_{11} = \frac{mg}{HTL} x_1 + C \]

On the left surface of the block \( n_i = (-1, 0, 0) \) and \( t_j^n = 0 \), i.e. no traction. Hence

\[ \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = -\sigma_{11} = 0 \]

So

\[ \sigma_{11}(x_1 = 0) = 0 \rightarrow C = 0 \]

so that

\[ \sigma_{11} = \frac{mg}{HT} \frac{x_1}{L} \]

It’s easy to see that at \( x_1 = L \), \( \sigma_{11} \) matches the boundary value. Stress along 1 is tensile everywhere.

Along 2 we proceed exactly as in the column example, to find

\[ \sigma_{22} = -\rho g x_2 \]

The minimum \( \sigma_{22} \) is at the bottom \( x_2 = H \), where \( \sigma_{22} = -\rho g H \). This value matches the boundary condition:

\[ r = \frac{m_{\text{block}} g}{\rho Vg} = \frac{\rho Vg}{\rho y L} = \rho g H \]

Stress along 2 is compressive everywhere.

Finally

\[ \sigma = 0 \quad \text{except} \quad \sigma_{11} = \frac{mg}{HT} \frac{x_1}{L}, \sigma_{22} = -\rho g x_2 \]

or we can write the stress tensor in the matrix form:

\[ \sigma = \begin{pmatrix} \frac{mg}{HT} \frac{x_1}{L} & 0 & 0 \\ 0 & -\rho g x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Note that the stress state changes from point to point, hence this is an example of non-uniform stress field.

Since there only 2 non-zero normal stresses in the body when all shear stresses are zero, this is an example of a biaxial stress field.

**S44.** Use the fact that \( R \) is orthogonal:

\[ R \cdot R^T = I \]

So that

\[ \det(R \cdot R^T) = 1 = \det R \det R^T = (\det R)^2 \Leftrightarrow \det R \pm 1 \]

We leave it without a proof that \( \det R = -1 \) corresponds to a change from the right-handed to the left-
handed CS, or vice versa, in addition to rotation. If the handedness is to be preserved, then
\[ \det \mathbf{R} = 1 \]

**S45.** The cross product of vectors \( \mathbf{a} = (a_1, a_2, a_3)^T \) and \( \mathbf{b} = (b_1, b_2, b_3)^T \) by definition is

\[
\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}
\]

where \( \mathbf{e}^i \) are the basis vectors. Expanding the determinant by the first row:

\[
\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \mathbf{e}^1 - (a_1 b_3 - a_3 b_1) \mathbf{e}^2 + (a_1 b_2 - a_2 b_1) \mathbf{e}^3
\]
or as a column vector

\[
\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_3 - a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}
\]

On the other hand:

\[
e_{ijk} a_j b_k = \begin{pmatrix} e_{123} a_2 b_3 + e_{132} a_3 b_2 \\ e_{231} a_3 b_1 + e_{213} a_1 b_3 \\ e_{312} a_1 b_2 + e_{321} a_2 b_1 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_3 - a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}
\]

So

\[
\mathbf{a} \times \mathbf{b} = e_{ijk} a_j b_k
\]

**S46.** This simply follows from the definition. To get from \( pqr \) to \( prq \) one has to swap two neighbouring indices. After that it is impossible to obtain the original sequence \( pqr \) by cyclic permutation: \( prq \rightarrow rqp \rightarrow qpr \rightarrow prq \neq pqr \). Hence the sign of all components of \( e_{pqr} \) will change.

**S47.** In 3D space 3 non-coplanar vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) define a parallelepiped the volume of which is

\[ V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \]

(BTW, explain why the meaning of \( \mathbf{a} \times (\mathbf{b} \cdot \mathbf{c}) = \mathbf{b} \times (\mathbf{c} \cdot \mathbf{a}) = \mathbf{c} \times (\mathbf{a} \cdot \mathbf{b}) \) is completely different.)

Using the index notation for the cross product (see Ex. prob. 45):

\[ V = e_{ijk} a_j b_j c_k \]

Note that this notation is superior to the \( \times \) notation because now the equation is completely symmetric wrt \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) and no brackets are needed.

Volume cannot change with the change in CS, hence

\[ V = e_{ijk} a_j b_j c_k = e_{ijk} a'_j b'_j c'_k \]

or using the vector rotation law:

\[ e_{ijk} R_{ip}^T a'_p R_{jq}^T b'_q R_{kr}^T c'_r = e_{ijk} a'_j b'_j c'_k \]

or swapping \( ijk \) and \( pqr \) on the left:

\[ e_{pqr} R_{ip}^T a'_i R_{jq}^T b'_j R_{kr}^T c'_k = e_{ijk} a'_i b'_j c'_k \]

or since \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) are arbitrary, we can cancel those from both sides:

\[ e_{pqr} R_{ip}^T R_{jq}^T R_{kr} = e_{ijk} \]

finally:
which is the definition of a rank 3 tensor. However, a simpler proof is to note that a volume is the same in any CS, hence, by definition, \( e_{ijk} \) is a rank 3 tensor.

**S48.** Starting from the rotation law for a rank 3 tensor:

\[
e_{ijk} = R_{ip} R_{jq} R_{kr} e_{pqr}
\]

Let’s see how the non-zero components transform, e.g. 123:

\[
e_{123}' = R_{1p} R_{2q} R_{3r} e_{pqr} = \det R = 1
\]

and the same clearly for any cyclic permutation. For the negative components, e.g. 132:

\[
e_{132}' = R_{1p} R_{3q} R_{2r} e_{pqr} = R_{1p} R_{2r} R_{3q} e_{pqr}
\]

or using \( e_{pqr} = -e_{pqr} \) (see ex. prob. 46):

\[
e_{132}' = R_{1p} R_{2r} R_{3q} (-e_{pqr}) = -\det R = -1
\]

Now let’s see how the zero components are transformed, e.g. 112:

\[
e_{112}' = R_{1p} R_{1q} R_{2r} e_{pqr}
\]

Note that for any \( r \) there are only two non-zero terms, with opposite signs, which cancel each other:

\[
R_{1p} R_{1q} R_{2r} - R_{1q} R_{1p} R_{2r} = 0
\]

The same is obviously true for any other combination with repeated indices, hence:

\[
e'_{ijk} = e_{ijk}
\]

**S49.** Writing \( T \) as a 3 x 3 symmetric matrix

\[
T = \begin{pmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{pmatrix}
\]

we can use row 1 to calculate the determinant:

\[
\det T = T_{11} \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} - T_{12} \begin{vmatrix} T_{21} & T_{23} \\ T_{31} & T_{33} \end{vmatrix} + T_{13} \begin{vmatrix} T_{21} & T_{22} \\ T_{31} & T_{32} \end{vmatrix}
\]

\[
= T_{11} (T_{22} T_{33} - T_{23}^2) - T_{12} (T_{21} T_{33} - T_{23} T_{31}) + T_{13} (T_{21} T_{32} - T_{22} T_{31})
\]

\[
= T_{11} T_{22} T_{33} + 2T_{12} T_{23} T_{31} - T_{11} T_{23}^2 - T_{22} T_{31}^2 - T_{33} T_{12}^2
\]

**S50.** Remember that the alternating tensor \( e_{ijk} = +1 \leftrightarrow ijk = 123, 231, 312, \) \( e_{ijk} = -1 \leftrightarrow ijk = 132, 213, 321, \) and zero otherwise. So

\[
e_{ijk} T_{11} T_{j2} T_{k3} = T_{11} T_{22} T_{33} + T_{21} T_{32} T_{13} + T_{31} T_{12} T_{23} - T_{11} T_{32} T_{23} - T_{31} T_{22} T_{13} - T_{21} T_{12} T_{33}
\]

\[
= T_{11} T_{22} T_{33} + 2T_{12} T_{23} T_{31} - T_{11} T_{23}^2 - T_{22} T_{31}^2 - T_{33} T_{12}^2
\]

Now compare the last expression to that from ex. prob. 49.

**S51.** Using \( \det T = e_{ijk} T_{11} T_{j2} T_{k3} \), see ex. prob. 50, we get from (86)

\[
\det (T - \lambda I) = (T_{11} - \lambda)(T_{22} - \lambda)(T_{33} - \lambda) + 2T_{12} T_{23} T_{31} - (T_{11} - \lambda) T_{23}^2 - (T_{22} - \lambda) T_{31}^2 - (T_{33} - \lambda) T_{12}^2
\]

\[
= -\lambda^3 + \lambda^2 T_{11} + \lambda^2 T_{22} + \lambda^2 T_{33} - \lambda T_{11} T_{22} - \lambda T_{22} T_{33} - \lambda T_{33} T_{11} + T_{11} T_{22} T_{33}
\]

\[
+2T_{12} T_{23} T_{31} - T_{11} T_{23}^2 - T_{22} T_{31}^2 - T_{33} T_{12}^2 + \lambda T_{11}^2 + \lambda T_{22}^2 + \lambda T_{33}^2
\]
$$\begin{align*}
= -\lambda^3 + \lambda^2(T_{11} + T_{22} + T_{33}) + \lambda(T_{12}^2 + T_{23}^2 + T_{31}^2 - T_{11}T_{22} - T_{22}T_{33} - T_{33}T_{11}) \\
+ T_{11}T_{22}T_{33} + 2T_{12}T_{23}T_{31} - T_{11}T_{23}^2 - T_{22}T_{31}^2 - T_{33}T_{12}^2
\end{align*}$$

or using the definition of $J^T$, Eqn. (88), and solution to ex. prob. 50:

$$\det (T - \lambda I) = -\lambda^3 + \lambda^2 T^T + \lambda(T_{12}^2 + T_{23}^2 + T_{31}^2 - T_{11}T_{22} - T_{22}T_{33} - T_{33}T_{11}) + III^T$$

To better understand the $\lambda$ term, let's expand $H^T$, from Eqn. (89):

$$H^T = T: T = T_{ij}T_{ij} = T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{12}^2 + 2T_{13}^2 + 2T_{23}^2$$

note factors 2 due to symmetry. Now let's see how $(I^T)^2$ look like:

$$(I^T)^2 = T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{11}T_{22} + 2T_{22}T_{33} + 2T_{33}T_{11}$$

We are looking to cancel the first 3 terms in the above two expressions, so that:

$$H^T - (I^T)^2 = 2T_{12}^2 + 2T_{23}^2 + 2T_{31}^2 - 2T_{11}T_{22} - 2T_{22}T_{33} - 2T_{33}T_{11}$$

Clearly the $\lambda$ term in the cubic equation above is a half of the last expression, so that

$$\det (T - \lambda I) = -\lambda^3 + \lambda^2 T^T + \lambda \frac{1}{2} (H^T - (I^T)^2) + III^T$$

**S52.** Remember that $R2T$ rotates with CT as

$$T' = R \cdot T \cdot R^T = T_{ij}' = R_{ik}T_{kp}R_{pj}' = R_{ik}R_{jp}T_{kp}$$

Also remember that $R$ is orthogonal, which means $R_{ik}R_{jk} = R_{ji}R_{kj} = \delta_{ij}$. So

$$tr(T') = T_{ii}' = R_{ik}R_{jp}T_{kp} = \delta_{kp}T_{kp} = T_{pp}$$

**S53.** As in ex. prob 52

$$T': T' = T_{ij}'T_{ij}' = R_{ik}R_{jp}T_{kl}R_{mn}R_{jm}T_{mn} = \delta_{km}\delta_{ln}T_{kl}T_{mn} = T_{kl}T_{kl}$$

**S54.** As in ex. prob. (88)

$$III^T = e_{ijk}T_{1i}'T_{2j}'T_{3k}' = R_{ip}R_{jq}R_{kp}e_{pqr}R_{uT_{1i}'R_{jp}T_{2j}'R_{ku}T_{3k}'}$$

Using the orthogonality of $R$:

$$III^T = \delta_{ip}\delta_{jq}\delta_{ku}e_{pqr}T_{1i}'T_{2j}'T_{3k}' = e_{stu}T_{1s}T_{2t}T_{3u}' = III^T$$

**S55.** Just rewrite the tensor using the principal values:

$$T = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}$$

Then

$$I^T = \lambda_1 + \lambda_2 + \lambda_3$$

$$H^T = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$III^T = \lambda_1\lambda_2\lambda_3$$

**S56.** Consider a rod under tension:
The rod of ∅d is loaded by force P. The average stress is therefore:
\[ \sigma = \frac{P}{A} = \frac{P}{\pi r^2} = \frac{4P}{\pi d^2} \]

If we are clever and choose the right handed CS as shown above then the stress tensor will be:
\[ \sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Hence \( I^\sigma = \sigma, \ II^\sigma = \sigma^2, \ III^\sigma = 0 \). The characteristic equation then gives \( \lambda_1 = \sigma, \ \lambda_2 = \lambda_3 = 0 \). Let’s check:
\[ \lambda_1 + \lambda_2 + \lambda_3 = \sigma = I^\sigma \]
\[ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \sigma^2 = II^\sigma \]
\[ \lambda_1 \lambda_2 \lambda_3 = 0 = III^\sigma \]

Let’s find the corresponding principal directions from Eqn. (85):

(1) \( \lambda_1 = \sigma \) gives:
\[
\begin{align*}
0 \cdot x_1 &= 0 \\
-\sigma x_2 &= 0 \\
-\sigma x_3 &= 0
\end{align*}
\]
So \( x_2 = x_3 = 0, \ x_1 = \pm 1 \). It is up to us to choose one. Let’s take +1. So the first principal vector is:
\[ x^1 = (1, 0, 0)^T \]

(2) \( \lambda_2 = 0 \) gives:
\[
\begin{align*}
\sigma x_1 &= 0 \\
0 &= 0 \\
0 &= 0
\end{align*}
\]
So that \( x_1 = 0, \) and \( x_2, x_3 \) are free. There is an infinite number of solutions.

(3) \( \lambda_3 = 0 \) gives exactly the same: \( x_1 = 0, \) and \( x_2, x_3 \) are free.

What does this mean? \( x^2, x^3 \perp (1, 0, 0)^T \). Can choose any values for free components, but! must make sure the basis vectors are orthogonal and the CS is right handed, e.g.:
\[ x^2 = (0, 1, 0)^T, \quad x^3 = (0, 0, 1)^T \]
or
\[ x^2 = (0, 0, 1)^T, \quad x^3 = (0, -1, 0)^T \]
The first solution corresponds to the CS drawn, which means it is already a principal CS. The second solution corresponds to the original CS rotated about axis 1 by \( \pi/2 \) clockwise if looking along it.

Now imagine that we were dumb, and chose this CS:

![Diagram of CS]

What would the stress tensor be in this CS? There are at least two solution methods. In the first method one can transform the stress tensor from the old CS to the new. In the second method one can define the stress tensor directly in the new CS. Let’s start with the first method.

The rotation tensor is

\[
R = \begin{pmatrix}
\cos \pi/4 & 0 & \cos \pi/4 \\
\cos \pi/4 & 0 & \cos 3\pi/4 \\
0 & 1 & 0
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{2} & 0 & 1/\sqrt{2} \\
1/\sqrt{2} & 0 & -1/\sqrt{2} \\
0 & 1 & 0
\end{pmatrix}
\]

The stress tensor in the new CS is \( \sigma’ = R\sigma R^T \), so:

\[
R\sigma = \begin{pmatrix}
1/\sqrt{2} & 0 & 1/\sqrt{2} \\
1/\sqrt{2} & 0 & -1/\sqrt{2} \\
0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\sigma & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\sigma/\sqrt{2} & 0 & 0 \\
\sigma/\sqrt{2} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
R\sigma R^T = \begin{pmatrix}
\sigma/\sqrt{2} & 0 & 0 \\
\sigma/\sqrt{2} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
1/\sqrt{2} & -1/\sqrt{2} & 0
\end{pmatrix} = \begin{pmatrix}
\sigma/2 & \sigma/2 & 0 \\
\sigma/2 & \sigma/2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Alternatively the stress tensor can be constructed directly in the new CS following the definitions of the components.

![Diagram of stress tensor components]

It is clear that \( \sigma’_{31} = \sigma’_{33} = 0 \). From geometry

\[
\sigma’_{11} = \sigma’_{12} = \sigma’_{21} = \sigma’_{22} = \frac{\sigma}{\sqrt{2}} \cos \frac{\pi}{4} = \sigma/2
\]

which matches the stress tensor components calculated via rotation.

The rotated tensor is symmetric, of course. We now have non-zero shear stress, and the stress state is biaxial. The invariants are: \( I^\sigma = \sigma/2 + \sigma/2 = \sigma \), \( II^\sigma = 4 \cdot \sigma/4 = \sigma^2 \). \( III^\sigma = 0 \). Note that these, of course, match the values found in the old CS. Clearly this means that the principal values are the same too. Of course; these are invariant to CT too! What about the principal directions? As before, using Eqn. (85):
(1) $\lambda_1 = \sigma$ gives:

$$\begin{cases}
(\sigma/2 - \sigma)x_1 + \sigma/2 x_2 = 0 \\
\sigma/2 x_1 + (\sigma/2 - \sigma)x_2 = 0 \\
-\sigma x_3 = 0
\end{cases}$$

So $x_3 = 0$. Subtracting the first 2 equations we get $x_1 = x_2$. The length of the basis vector

$$\mathbf{x} \cdot \mathbf{x} = x_i x_i = 1 = x_1^2 + x_2^2 + x_3^2$$

hence $x_1^2 + x_2^2 = 1$, or $x_1 = x_2 = \pm 1/\sqrt{2}$. We can choose any valid combination, say:

$$\mathbf{x}^1 = (1/\sqrt{2}, 1/\sqrt{2}, 0)^T$$

(2) $\lambda_2 = 0$ gives:

$$\begin{cases}
\frac{\sigma}{2} x_1 + \frac{\sigma}{2} x_2 = 0 \\
\frac{\sigma}{2} x_1 + \frac{\sigma}{2} x_2 = 0 \\
0 = 0
\end{cases}$$

So that $x_1 = -x_2$, $x_3$ is free. We can arbitrarily set $x_1 = x_2 = 0$, $x_3 = 1$, so that:

$$\mathbf{x}^2 = (0, 0, 1)^T$$

It's easy to check that $\mathbf{x}^1 \perp \mathbf{x}^2$.

(3) $\lambda_3 = 0$ gives as before: $x_1 = -x_2$, $x_3$ is free. Since the first two new basis vectors have been chosen already, we are restricted in our choice of the components of the third, because of the orthonormality constraints:

$$\mathbf{x}^3 \cdot \mathbf{x}^1 = 0 \rightarrow x_1^3 \frac{1}{\sqrt{2}} + x_2^3 \frac{1}{\sqrt{2}} + x_3^3 \cdot 0 = 0 \rightarrow x_1^3 = -x_2^3$$

$$\mathbf{x}^3 \cdot \mathbf{x}^2 = 0 \rightarrow x_1^3 \cdot 0 + x_2^3 \cdot 0 + x_3^3 \cdot 1 = 0 \rightarrow x_3^3 = 0$$

$$\mathbf{x}^3 \cdot \mathbf{x}^3 = 1 \rightarrow x_1^3 = -x_2^3 = \pm \frac{1}{\sqrt{2}}$$

Finally let's choose one of the valid combinations:

$$\mathbf{x}^3 = (1/\sqrt{2}, -1/\sqrt{2}, 0)^T$$

Let's draw the principal CS:

---

S57. The invariants are:

$I^\sigma = 300$
\[ II^\sigma = 10^0 + 300^2 + 100^2 + 2 \times 200^2 + 2 \times 200^2 + 2 \times 400^2 = 59 \times 10^4 \]

\[ III^\sigma = -100 \begin{vmatrix} 300 & 400 \\ 400 & 100 \end{vmatrix} - 200 \begin{vmatrix} 200 & 400 \\ -200 & 100 \end{vmatrix} - 200 \begin{vmatrix} 200 & 300 \\ -200 & 400 \end{vmatrix} = 13 \times 10^6 - 20 \times 10^6 - 28 \times 10^6 = -35 \times 10^6 \]

So that

\[ \frac{1}{2} (II^\sigma - (I^\sigma)^2) = \frac{1}{2} (59 \times 10^4 - 9 \times 10^4) = 25 \times 10^4 \]

From Eqn. (86) the characteristic equation looks like

\[ -\lambda^3 + 300\lambda^2 + 25 \times 10^4 \lambda - 35 \times 10^6 = 0 \]

I solve this problem numerically using LAPACK library, routine DSYEV, with Fortran.

Original tensor

\[
\begin{array}{ccc}
-1.00000000000E+02 & 2.00000000000E+02 & -2.00000000000E+02 \\
2.00000000000E+02 & 3.00000000000E+02 & 4.00000000000E+02 \\
-2.00000000000E+02 & 4.00000000000E+02 & 1.00000000000E+02
\end{array}
\]

Invariants:

\[
3.00000000000E+02 \quad 5.90000000000E+05 \quad -3.50000000000E+07
\]

The DSYEV eigenvalues in increasing order are:

\[
L3 \quad L2 \quad L1
\]

\[
-4.42895748878E+02 \quad 1.28655464368E+02 \quad 6.14240284510E+02
\]

DSYEV invariants check:

\[
3.00000000000E+02 \quad 5.90000000000E+05 \quad -3.50000000000E+07
\]

The DSYEV orthonormal eigenvectors (columns) are:

\[
x3 \quad x2 \quad x1
\]

\[
-6.34587302398E-01 \quad -7.70838350041E-01 \quad -5.57422078993E-02 \\
4.91831418220E-01 \quad -3.47155034107E-01 \quad -7.98489347672E-01 \\
-5.96155023200E-01 \quad 5.34126970299E-01 \quad -5.99422695527E-01
\]

The angles (deg) are:

\[
1.29389382175E+02 \quad 1.40429231753E+02 \quad 9.31954495260E+01 \\
6.05389733835E+01 \quad 1.10313402693E+02 \quad 1.42986086857E+02 \\
1.26595014954E+02 \quad 5.77152759561E+01 \quad 1.26828562438E+02
\]

So \( \lambda_1 = 614, \lambda_2 = 129, \lambda_3 = -443 \), \( x^1 = (-0.0557, -0.798, -0.599) \), \( x^2 = (-0.771, -0.347, -0.534) \) and \( x^3 = (-0.635, 0.492, -0.596) \). The principal CS can be drawn based on the angles above.

The initial stress state looks like this:

The principal stress state looks like this:
S58. Start from the definition of $F$:

$$F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

In another CS:

$$F' = \frac{\partial \mathbf{x}'}{\partial \mathbf{X}'}$$

By definition:

$$\mathbf{x}' = R \cdot \mathbf{x} \quad ; \quad \mathbf{X}' = R \cdot \mathbf{X}$$

Using chain rule:

$$F' = \frac{\partial \mathbf{x}'}{\partial \mathbf{X}'} = \frac{\partial \mathbf{x}'}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{X}'} = \frac{\partial (R \cdot \mathbf{x})}{\partial \mathbf{X}} \cdot \frac{\partial (R^T \cdot \mathbf{X}')}{\partial \mathbf{X}'}$$

This is one of the definitions of $R^2T$.

S59. Start from Eqn. (103). We’ll use the index notation, as it is more explicit:

$$E_{jk} = \frac{1}{2} (F_{ij}F_{ik} - \delta_{jk})$$

or using Eqn. (95):

$$E_{jk} = \frac{1}{2} \left( \frac{\partial x_i}{\partial X_j} \frac{\partial x_i}{\partial X_k} - \delta_{jk} \right)$$

or using

$$x_i = X_i + u_i$$

we obtain

$$E_{jk} = \frac{1}{2} \left( \frac{\partial (X_i + u_i)}{\partial X_j} \frac{\partial (X_i + u_i)}{\partial X_k} - \delta_{jk} \right)$$

$$= \frac{1}{2} \left( (\delta_{ij} + \frac{\partial u_i}{\partial X_j})(\delta_{ik} + \frac{\partial u_i}{\partial X_k}) - \delta_{jk} \right)$$

$$= \frac{1}{2}(u_{j,k} + u_{k,j} + u_{i,j}u_{i,k})$$

S60. Two proofs are possible. Proof 1 is based on first proving that $F$ is $R^2T$ (ex. prob. 58). Then use Eqn. (103):

$$E = \frac{1}{2}(F^T \cdot F - I)$$

In another CS:
\[ E' = \frac{1}{2}(F'^T \cdot F' - I) \]

Remember that \( I' = I \). Given that \( F \) is R2T:

\[ E' = \frac{1}{2}(F'^T \cdot F' - I) = \frac{1}{2}((R \cdot F \cdot R^T)^T \cdot R \cdot F \cdot R^T - I) = \frac{1}{2}(R \cdot F^T \cdot R^T \cdot R \cdot F \cdot R^T - I) \]

\[ R \cdot R^T = I \]

so

\[ = \frac{1}{2}(R \cdot F^T \cdot F \cdot R^T - I) = R \cdot \frac{1}{2}(F^T \cdot F - I) \cdot R^T = R \cdot E \cdot R^T \]

Here we used \( I = R \cdot I \cdot R^T \).

The second proof proceeds similarly to that in ex. prob. 58. We start from (104):

\[ E_{jk}' = \frac{1}{2}(u_{i,j}' + u_{j,i}' + u_{k,j}'u_{k,j}') \]

To prove that \( E_{ij} \) is R2T, it is sufficient to prove that \( u_{i,j} \) is R2T.

We use \( u_{i}' = R_{ij}u_j, \ x_{i}' = R_{ij}x_j \). Using the chain rule:

\[ u_{i,j}' = \frac{\partial u_{i,j}'}{\partial x_j} = \frac{\partial(R_{ik}u_k)}{\partial x_j} \frac{\partial x_j}{\partial x_j}' = R_{ik}u_k \frac{\partial(R_{pm}x_m')}{\partial x_j}' = R_{ik}u_k \delta_{mj}R_{pm} = R_{ik}u_{k,p}R_{pj} \]

which is one of the definitions of R2T. In tensor notation

\[ \nabla u' = R \cdot \nabla u \cdot R^T \]

**S61.** Written explicitly the motion is:

\[ x_1 = X_1 + tX_2 \]
\[ x_2 = X_2 \]
\[ x_3 = X_3 \]

The motion can be drawn like this:

![Diagram](image)

We can use just a 2D drawing, because there is no motion along direction 3. What is shown in an undeformed unit cube of material (solid lines), and its deformed shape (dashed lines). Note that the motion and the deformation are *linear*, meaning that straight lines stay straight throughout the deformation.

Let’s calculate \( F \):

\[ F = \frac{\partial x}{\partial X} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

Note that the deformation gradient is not symmetric.

The strain tensor is:

\[ E = \frac{1}{2}(F^T \cdot F - I) \]

First
\[
\mathbf{F}^T \cdot \mathbf{F} = \begin{pmatrix}
1 & 0 & 0 \\
t & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & t & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & t & 0 \\
t & t^2 + 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

So that
\[
\mathbf{E} = \begin{pmatrix}
0 & t/2 & 0 \\
t/2 & t^2/2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Note that the strain tensor is symmetric, unlike the deformation gradient.

Alternatively we can calculate strain from displacement field:
\[
u_1 = x_1 - X_1 = tX_2
\]
\[
u_2 = x_2 - X_2 = 0
\]
\[
u_3 = x_3 - X_3 = 0
\]

So that
\[
E_{ij} = \frac{1}{2}(\nu_{i,j} + \nu_{j,i} + \nu^T_{i,k} \nu_{k,j})
\]

First
\[
u_{i,j} = \begin{pmatrix}
0 & t & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

So that
\[
\nu^T_{i,k} \nu_{k,j} = \begin{pmatrix}
0 & 0 & 0 \\
t & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
t & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & t^2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Finally
\[
E_{ij} = \frac{1}{2}
\begin{pmatrix}
0 & t & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 \\
0 & t & 0 \\
0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 \\
0 & t^2 & 0 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & t/2 & 0 \\
t/2 & t^2/2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
as before.

This example makes the distinction between the small and the finite strain formulations very clear. Let’s assume that the strains stay below 1%, which is typical for elastic strains in engineering structures. That means that \( t = 10^{-2} \), which means that \( t^2 = 10^{-4} \), i.e. two orders of magnitude smaller. Hence, in this case the \( t^2 \) term can be neglected, and the small strain tensor will be:
\[
\mathbf{e} = \begin{pmatrix}
0 & t/2 & 0 \\
t/2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Note that now there are no normal strains, only shear. This means that the lengths of line elements don’t change, only the angle between them. If you look at the drawing and imagine that \( t \) is small, you will see that this is correct. Indeed if \( t \) is small, then \( t = \tan \alpha = \alpha \). Therefore small scale pure shear deformation is simply change of angles between straight lines. Moreover, if the deformation is redrawn symmetrically wrt the original shape, the meaning of \( \frac{1}{2} \) factors becomes clear:
In contrast, if $t$ is not small compared to 1, as is the case drawn above, then the $t^2$ term is non-negligible and cannot be dropped. Indeed, for rubber-like materials $t$ can be many times greater than 1, giving a very large $t^2$ term, meaning that $\varepsilon_{22}$, the elongation along the initial direction 2 will be large. Again, use the drawing to visualise this.

This deformation type is called simple shear or pure shear.

Finally, $I^e = 0$, hence the deformation is incompressible.

**S62.** The invariants are: $I^e = 0$, $II^e = t^2/2$, $III^e = 0$. The characteristic equation, Eqn. (87), will look like this:

$$-\varepsilon^3 + \frac{t^2}{4} \varepsilon = 0$$

The roots in decreasing order are: $\varepsilon_1 = t/2$, $\varepsilon_2 = 0$, $\varepsilon_3 = -t/2$, and the strain tensor in the principal directions will look like this:

$$\varepsilon = \begin{pmatrix} t/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -t/2 \end{pmatrix}$$

The principal directions are found from Eqn. (85).

1. $\lambda_3 = -t/2$:

$$\frac{t}{2} x_1 + \frac{t}{2} x_2 = 0$$

$$\frac{t}{2} x_1 + \frac{t}{2} x_2 = 0$$

$$\frac{t}{2} x_3 = 0$$

The solution is $x_1 = -x_2$, $x_3 = 0$. We can choose any solution, as long as $x_i \otimes x_i = 1$ (or in index notation $x'_k x'_k = \delta_{ij}$, where $x'_k$ is $k$th component of $i$th principal (basis) vector). Let’s choose $x^3 = (\sqrt{2}/2, -\sqrt{2}/2, 0)$.

2. $\lambda_2 = 0$:

$$\frac{t}{2} x_2 = 0$$

$$\frac{t}{2} x_1 = 0$$

$$0 x_3 = 0$$

The solution is $x_1 = x_2 = 0$, $x_3 = \pm 1$. Again, we are free to choose either solution. Let’s choose $x^2 = (0, 0, 1)$.

3. $\lambda_1 = t/2$: 
\[-\frac{t}{2} x_1 + \frac{t}{2} x_2 = 0\]
\[\frac{t}{2} x_1 - \frac{t}{2} x_2 = 0\]
\[\frac{t}{2} x_3 = 0\]

The solution is \(x_1 = x_2, x_3 = 0\). Since the other 2 vectors have been set already, we have no freedom left. Provided we want to keep the CS right-handed we must choose: \(x^1 = (\sqrt{2}/2, \sqrt{2}/2, 0)\). Check that \(x^1 \cdot x^3 = 0\) for orthonormality.

We can now draw the deformation in principal coordinates:

Note that in the principal coordinates angles between any pairs of straight lines stay constant. In contrast the lengths change.

This drawing illustrates well that the deformation is incompressible. The volume of the undeformed unit cube is \(V = 1\). The volume of the deformed cube is
\[v = (1 + t/2)(1 - t/2) = 1 - t^2/4\]
If \(t\) is small, i.e. the small strain theory is assumed, then \(v = 1 = V\), which is the meaning of incompressible deformation.

S63. The rotation tensor corresponds to rotating about axis 3 by \(\pi/2\) clockwise. This can be drawn as:

Let's first calculate \(F'\) directly:
\[F' = \frac{\partial x'}{\partial X'}\]
where
\[x' = R \cdot x\]
\[X' = R \cdot X\]

Calculating:
Check on the drawing that these expressions are correct.

Similarly

Combining the last 2 lines:

Again check on the drawing that these expressions are correct.

Finally with simple differentiation:

To verify that $F$ is R2T, just need to do 2 matrix multiplications:

Finally

which agrees with the answer obtained directly, hence $F$ is indeed R2T.

**S64.** We assume a popular rosette where the three strain gauges are oriented at 0°, 45° and 90°:

We can accept the orientations of 0° and 90° gauges as a natural choice of CS.

We denote strains measured along these directions as $\varepsilon_0$, $\varepsilon_{45}$ and $\varepsilon_{90}$. With that $\varepsilon_{11} = \varepsilon_0$, $\varepsilon_{22} = \varepsilon_{90}$.

All we need to do is to resolve $\varepsilon_{45}$ into our CS.

The rotation tensor that transforms the 12 CS into the 1′2′ CS is

$$ R = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} $$
In 12 CS the strain tensor looks like:

\[ \varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{21} & \varepsilon_{31} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{32} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \]

In vast majority of cases the strain gauge is applied to the free surface, i.e. there are no surface tractions. This means \( \varepsilon_{13} = \varepsilon_{23} = 0 \). With this the strain tensor will be

\[ \varepsilon = \begin{pmatrix} \varepsilon_0 & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{90} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix} \]

The strain gauge rosette measurement says nothing about \( \varepsilon_{33} \). This remains unknown and has to be found from other experiments. However, we know that this strain is unchanged by rotating CS about axis 3. This allows us finding \( \varepsilon_{12} \) without knowing \( \varepsilon_{33} \).

So

\[ \varepsilon_{45} = \varepsilon_{11}' = R_{1i} \varepsilon_{ij} R_{ji} = \begin{pmatrix} \sqrt{2}/2 & 0 \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \varepsilon_0 + \varepsilon_{12} \\ \sqrt{2}/2 \varepsilon_{12} + \varepsilon_{90} \end{pmatrix} \]

\[ = \frac{1}{2}(\varepsilon_0 + 2 \varepsilon_{12} + \varepsilon_{90}) \]

From where

\[ \varepsilon_{12} = \varepsilon_{45} - \frac{1}{2}(\varepsilon_0 + \varepsilon_{90}) \]

**S65.** Let’s work with a stress tensor. Exactly the same analysis applies to the strain tensor, just swap \( \sigma \) for \( \varepsilon \).

Start from the tensor is principal directions:

\[ \sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \]

Next we need to understand how shear stress is related to other components of stress tensor. Refer to the diagram

An element of surface with normal \( n \) has the stress vector \( t \), which can be decomposed into the normal, \( p \), and the shear (tangential), \( s \) vectors:

\[ s = t - p \]

where using Eqn. (63) and the fact that \( \sigma \) is diagonal:
Using the chain rule:

\[ \frac{df}{dn} = \frac{\partial f}{\partial n_1} dn_1 + \frac{\partial f}{\partial n_2} dn_2 + \frac{\partial f}{\partial n_3} dn_3 + \frac{df}{L} dL \]

Note that the last term simply recovers the constraint. \( n_1, n_2 \) and \( n_3 \) are treated as three independent variables (subject only to the above constraint). This means that for \( df = 0 \), all three partial derivatives above must vanish:

\[ \frac{\partial f}{\partial n_1} = 0 \quad ; \quad \frac{\partial f}{\partial n_2} = 0 \quad ; \quad \frac{\partial f}{\partial n_3} = 0 \]

Let’s find these derivatives. For this we need to calculate \( s \cdot s \) first:

\[ s \cdot s = (\sigma_1 - z)^2 n_1^2 + (\sigma_2 - z)^2 n_2^2 + (\sigma_3 - z)^2 n_3^2 \]

\[ = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 + z^2 n_1^2 + z^2 n_2^2 + z^2 n_3^2 - 2\sigma_1 zn_1^2 - 2\sigma_2 zn_2^2 - 2\sigma_3 zn_3^2 \]

\[ = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 + z^2 (n_1^2 + n_2^2 + n_3^2) - 2z(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2) \]

\[ = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 + z^2 - 2z^2 \]

\[ = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2 \]

With that:
\[
\frac{\partial f}{\partial n_1} = \frac{\partial(s \cdot s)}{\partial n_1} + 2Ln_1 = 0
\]

or

\[
2n_1\sigma_1^2 - 2z2\sigma_1n_1 + 2Ln_1 = 0
\]

This equation splits into 2 branches:

\[
n_1 = 0 \quad \text{or} \quad \sigma_1^2 - 2z\sigma_1 + L = 0 \quad (148)
\]

Because all expressions are symmetrical wrt \(\sigma\) and \(n\), we can straight away write the other 2 equations:

\[
n_2 = 0 \quad \text{or} \quad \sigma_2^2 - 2z\sigma_2 + L = 0 \quad (149)
\]

\[
n_3 = 0 \quad \text{or} \quad \sigma_3^2 - 2z\sigma_3 + L = 0 \quad (150)
\]

Let’s first consider the cases where two projections of the normal are zero. For example, if \(n_1 = n_2 = 0\), then from the unity constraint \(n_3 = \pm 1\). However, we immediately see that this direction is one of the principal directions, where \(s = 0\). This can also be checked from the expression for \(s \cdot s\). So these directions give the orientations on which shear stress magnitude is minimum i.e. zero.

Next let’s explore cases when only one component of \(n\) is zero, e.g. \(n_1 = 0, n_2 \neq 0, n_3 \neq 0\). In this case Eqns. (149) and (150) will be:

\[
\sigma_2^2 - 2z\sigma_2 + L = 0
\]

\[
\sigma_3^2 - 2z\sigma_3 + L = 0
\]

By subtracting one from the other we get rid of \(L\):

\[
\sigma_2^2 - \sigma_3^2 + 2z(\sigma_3 - \sigma_2) = 0
\]

or

\[
(\sigma_2 - \sigma_3)(\sigma_2 + \sigma_3) + 2z(\sigma_3 - \sigma_2) = 0
\]

Again we have 2 branches:

\[
\sigma_2 = \sigma_3 \quad \text{or} \quad \sigma_2 + \sigma_3 - 2z = 0
\]

Let’s follow the second branch now. Using the expression for \(z\), with \(n_1 = 0\), we obtain:

\[
\sigma_2 + \sigma_3 - 2\sigma_2n_2^2 - 2\sigma_3n_3^2 = 0
\]

From the unity constraint: \(n_2^2 = 1 - n_3^2\). With that the above equation is rewritten as:

\[
\sigma_2 + \sigma_3 - 2\sigma_2(1 - n_3^2) - 2\sigma_3n_3^2 = 0
\]

or

\[
\sigma_3 - \sigma_2 + 2n_3^2(\sigma_2 - \sigma_3) = 0
\]

Since in this branch \(\sigma_2 \neq \sigma_3\), then

\[
2n_3^2 = 1
\]

or

\[
n_3 = \pm \sqrt{2}/2
\]

hence from the orthonormality constrain

\[
n_2 = \pm \sqrt{2}/2
\]

but with the opposite sign to \(n_3\). One solution is thus \(n = (0, \pm \sqrt{2}/2, \mp \sqrt{2}/2)\). The factors are sin and cos of 45°. So this vector is in 23 plane and makes 45° angles with both 2 and 3 axes.
What is the magnitude of this extremum shear stress? We put the found \( \mathbf{u} \) into the expression for \( \mathbf{s} \cdot \mathbf{s} \):

\[
\mathbf{s} \cdot \mathbf{s} = \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} - \frac{(\sigma_2^2 + \sigma_3^2)}{2} = \frac{(\sigma_2 - \sigma_3)^2}{2}
\]

Hence the magnitude of this shear value is

\[
|\mathbf{s}| = \frac{\sigma_2 - \sigma_3}{2}
\]

Following exactly the same logic for the other 2 cases, when \( n_2 = 0 \) and \( n_3 = 0 \), we obtain these directions and extreme values:

\[
\mathbf{n} = \left( \pm \frac{\sqrt{2}}{2}, 0, \mp \frac{\sqrt{2}}{2} \right) ; \quad |\mathbf{s}| = \frac{\sigma_1 - \sigma_3}{2}
\]

\[
\mathbf{n} = \left( \pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2}, 0 \right) ; \quad |\mathbf{s}| = \frac{\sigma_1 - \sigma_2}{2}
\]

Given our convention that \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \), the absolute maximum shear stress is

\[
\tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2}
\]

It remains to check the other solution branches, for example \( n_1 = 0 \) and \( \sigma_2 = \sigma_3 \). By inserting these conditions into the \( \mathbf{s} \cdot \mathbf{s} \) expressions we obtain:

\[
\mathbf{s} \cdot \mathbf{s} = \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} - \frac{(\sigma_2^2 + \sigma_3^2)}{2} = \sigma_2^2 - \sigma_2^2 = 0
\]

So shear stresses on all planes parallel to \( n_1 \) are zero. Stress states with any 2 principal stresses equal are called equi-biaxial. The distinguishing feature of these stress states is that there is no shear stress in the equi-biaxial plane.

Finally we need to see if there are any solutions with \( n_1 \neq 0 \), \( n_2 \neq 0 \) and \( n_3 \neq 0 \). For this we subtract three pairs of Eqns.: (148) - (149), (149) - (150), (150) - (148):

\[
\sigma_1^2 - \sigma_2^2 + 2z(\sigma_2 - \sigma_1) = 0
\]

\[
\sigma_2^2 - \sigma_3^2 + 2z(\sigma_3 - \sigma_2) = 0
\]

\[
\sigma_3^2 - \sigma_1^2 + 2z(\sigma_1 - \sigma_3) = 0
\]

Solution is possible only iff

\[
\sigma_1 = \sigma_2 = \sigma_3
\]

which is called hydrostatic stress state. It is easy to see from the expression for \( \mathbf{s} \cdot \mathbf{s} \) that in this case shear stress is zero on any plane, which is the distinguishing feature of the hydrostatic stress state.

If the principal stresses are different, then one obtains 3 expressions for \( z \) from 3 equations above:

\[
2z = \frac{\sigma_2^2 - \sigma_1^2}{\sigma_2 - \sigma_1} = \sigma_2 + \sigma_1
\]

\[
2z = \frac{\sigma_3^2 - \sigma_2^2}{\sigma_3 - \sigma_2} = \sigma_3 + \sigma_2
\]

\[
2z = \frac{\sigma_1^2 - \sigma_3^2}{\sigma_1 - \sigma_3} = \sigma_1 + \sigma_3
\]

However, since the left hand sides are the same, all stresses must be equal, which is a contradiction. We conclude that there are no more possible solutions.

S66. Stress state:
These are also the only 3 planes with zero shear stress.

Maximum shear stress planes:

These are also three of the zero shear planes. However, other zero shear planes exist too.

Indeed, since $\varepsilon_1 = \varepsilon_3$, any plane parallel to direction 2 is zero shear. The strain state is equi-biaxial and 13 is the equi-biaxial plane.

Maximum shear strain planes:

**S67.** The stress tensor in the matrix form looks like this:
First need to recognise that $\sigma_{12}$ is at maximum when all normal stresses are zero. If it were not, then the Mohr’s diagram shows that there would have to be a non-zero normal stress:

So the diagram must look like this:

The maximum shear is the radius of the big circle. Since it is centred at the origin, the two principal stresses are immediately $\sigma_3 = -200\text{MPa}$, and $\sigma_1 = 200\text{MPa}$. The remaining principal stress $\sigma_2 = 0\text{MPa}$. That fits the radii and the centres of the two smaller circles.

We can now use the diagram. What will happen if we rotate the cube about axis 3 so that $\sigma_{12}$ drops to $100\text{MPa}$?

From geometry: $\cos 2\alpha = 100/200 = 0.5 \rightarrow 2\alpha = 60^\circ \rightarrow \alpha = 30^\circ$. The normal stresses are $200 \sin 2\alpha = 100\sqrt{3} = 173\text{MPa}$. The stress tensor will look like:

Note that extra sign conventions are needed to be sure what signs to use. Clearly the signs will flip if the sign of $\alpha$ will change.

If $\alpha = 45^\circ$, then we get to the principal state:
From the principal stress state we can now rotate about another axis, say 1. If $\alpha = 15^\circ$, then the stress state will look like this:

The shear stress $\sigma_{23} = 100 \sin 2\alpha = 50\text{MPa}$. The distance of normal stresses from the centre of this circle is $100 \cos 2\alpha = 87\text{MPa}$. With that the smaller normal stress is $100 - 87 = 13\text{MPa}$ and the larger normal stress is $100 + 87 = 187\text{MPa}$. Written as a matrix the stress state is:

$$\sigma = \begin{bmatrix} \pm 200 & 0 & 0 \\ 0 & \mp 200 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where we arbitrarily set the signs to match the Mohr’s diagram.

A warning, one more time: for the Mohr’s diagram to make any sense, one can rotate only about a single principal axis at a time, relative to the principal orientations. If more complex analysis is required, then the Mohr’s circle must not be used.

**S68.** The stress tensor:

The strain tensor:
Note that one circle shrunk to a point. Hence the two remaining circles coincide. This is a direct consequence of having two equal principal strains. No shear strains exist in the plane of those two principal directions, irrespective of any rotation about the other principal direction.

**S69.** Note that since \( \varepsilon_{13} = \varepsilon_{23} = 0 \) we conclude that \( \varepsilon_{33} \) is the principal strain. Hence we are only interested in 12 plane, and we can use a two-dimensional Mohr’s diagram, i.e. just a single circle.

The centre of the circle is at \( \frac{1}{2}(\varepsilon_{11} + \varepsilon_{22}) = -10^{-2} \). The radius of the circle, \( \gamma_{\text{max}} \), - the maximum shear strain, is calculated from the right angled triangle:

\[
\gamma_{\text{max}}^2 = \left(\frac{\varepsilon_{11} - \varepsilon_{22}}{2}\right)^2 + \varepsilon_{12}^2 = 8 \times 10^{-4}
\]

so that \( \gamma_{\text{max}} = 2\sqrt{2} \times 10^{-2} = 0.0283 \). With that the Mohr’s diagram can be drawn:

So the principal strains are \(-3.83 \times 10^{-2}\) and \(1.83 \times 10^{-2}\). Because we don’t know the value of the third principal strain, we cannot give these values definite label.

Finally note that we have obtained a general expression for principal values in two-dimensional cases:

\[
\gamma_1, \gamma_2 = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} \pm \sqrt{\left(\frac{\varepsilon_{11} - \varepsilon_{22}}{2}\right)^2 + \varepsilon_{12}^2}
\]

which is nothing more than a solution to a quadratic equation, which is what the characteristic equation is reduced to in 2D case.

**S70.** Symmetry of stress tensor means \( \sigma_{st} = \sigma_{ts} \). Hence one can swap subscripts \( t \) and \( s \) in Eqn. (109):

\[
\sigma_{st} = C_{\text{stab}} \varepsilon_{ab} = C_{\text{stab}} \varepsilon_{ba} = \sigma_{st}
\]

which means that

\[
C_{\text{stab}} = C_{\text{tsab}}
\]

This is called minor symmetry of \( \textbf{C} \). We will show later that rank 4 tensors can also have major symmetry.

There are thus three redundant combinations of \( st \): \( C_{12ab} = C_{21ab}, C_{23ab} = C_{32ab} \) and \( C_{31ab} = C_{13ab} \). Each of these equations has \( 3^2 = 9 \) unique combinations of \( ab \). Hence the total number of redundant components of \( \textbf{C} \) is \( 3 \times 9 = 27 \). This reduces the number of required constants in \( \textbf{C} \) from \( 3^4 = 81 \) to 54.

Exactly the same analysis can be made regarding strain: \( \varepsilon_{ab} = \varepsilon_{ba} \). Hence

\[
C_{\text{stab}} = C_{\text{tsba}}
\]

which is another minor symmetry in \( \textbf{C} \). There are 3 redundant combinations of \( ab \): \( C_{rst2} = C_{sr21}, C_{rst3} = C_{str3} \) and \( C_{rst1} = C_{str1} \). Each of these combinations has 6 unique combinations of \( st \). We say 6 and not 9 because we already have taken the symmetry of \( \sigma \) into account. The total number of redundant components of \( \textbf{C} \) is thus \( 3 \times 6 = 18 \). This further reduces the number of required independent components in \( \textbf{C} \) from 54 to 36.
Both minor symmetries in $\mathbf{C}$ can be summarised as:

$$\mathbf{C}_{\text{stab}} = \mathbf{C}_{\text{tsab}} = \mathbf{C}_{\text{tsba}} = \mathbf{C}_{\text{stba}}$$

The above discussion can be equally applied to the compliance tensor $\mathbf{S}$ to yield:

$$\mathbf{S}_{\text{stab}} = \mathbf{S}_{\text{tsab}} = \mathbf{S}_{\text{tsba}} = \mathbf{S}_{\text{stba}}$$

\textbf{S71.} From Eqn. (110) \textit{elastic energy density} can be calculated as

$$W = \int \sigma_{ab} d\varepsilon_{ab}$$

The units of $W$ are pressure, e.g. MPa, which is the same as energy per volume.

Using elasticity this can be rewritten as

$$W = \int C_{\text{abst}} \varepsilon_{st} d\varepsilon_{ab} = \frac{1}{2} C_{\text{abst}} \varepsilon_{ab} \varepsilon_{st}$$

or swapping $st$ and $ab$ we get:

$$W = \frac{1}{2} C_{\text{stab}} \varepsilon_{st} \varepsilon_{ab}$$

which means

$$C_{\text{stab}} = C_{\text{abst}}$$

This property is called the \textit{major symmetry}.

After taking the symmetry of $\sigma$ and $\varepsilon$ into account we now have 6 unique combinations of $st$ and 6 unique combinations of $ab$. Excluding 6 combinations of $ab = st$, we have $6 \times 6 - 6 = 30$ combinations of $stab$, half of which are redundant due to major symmetry. Finally $\mathbf{C}$ has $15 + 6 = 21$ independent components.

Exactly the same logic can be applied to $\mathbf{S}$ to show that

$$\mathbf{S}_{\text{stab}} = \mathbf{S}_{\text{abst}}$$

\textbf{S72.} Choose an arbitrary stress component, say $\sigma_{23}$. Express it as a function of stress with Eqn. (109). It will involve a double summation over both strain subscripts - $3 \times 3 = 9$ terms in total:

$$\sigma_{23} = C_{2311} \varepsilon_{11} + C_{2322} \varepsilon_{22} + C_{2333} \varepsilon_{33} + C_{2312} \varepsilon_{12} + C_{2321} \varepsilon_{21} + C_{2332} \varepsilon_{32} + C_{2323} \varepsilon_{23} + C_{2331} \varepsilon_{31} + C_{2313} \varepsilon_{13}$$

due to symmetry in strain $\varepsilon_{12} = \varepsilon_{21}$ etc., so:

$$\sigma_{23} = C_{2311} \varepsilon_{11} + C_{2322} \varepsilon_{22} + C_{2333} \varepsilon_{33} + 2C_{2312} \varepsilon_{12} + 2C_{2321} \varepsilon_{21} + 2C_{2332} \varepsilon_{32} + 2C_{2323} \varepsilon_{23} + 2C_{2331} \varepsilon_{31}$$

When written in the matrix form, these factors of 2 can be made a part of the elastic matrix, as in Eqn. (114), or a part of the strain vector, as in (115).

\textbf{S73.} First we need to explain the meaning of the Poisson’s ratio, $\nu$. Consider a uniaxial tension of a rod.

The dashed lines show the original, undeformed, shape of the rod. Solid lines show the shape of the rod after some tensile loading was applied at one end. The other end is considered constrained in this illustration.
Experiments show that as the rod elongates its cross section is reduced, although there are some very exotic artificial materials which show an increase of the cross section with rod elongation. The Poisson’s ratio quantifies the cross section change relative to change in length.

If axis 1 is along the axis of the rod and axes 2 and 3 are in the cross section, then these clearly are the principal strain axes. The strain tensor in these coordinates will look like:

\[
\varepsilon = \begin{pmatrix}
\varepsilon_{11} & 0 & 0 \\
0 & \varepsilon_{22} & 0 \\
\text{sym} & \varepsilon_{33}
\end{pmatrix}
\]

By definition, the Poisson’s ratio is

\[
\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}}
\]

In isotropic materials there is only a single Poisson’s ratio. This definition means that a positive Poisson’s ratio means that material will shrink in the directions normal to the pulling direction. And inversely, under uniaxial compression, the material will bulge out in the directions normal to the compression direction, if the Poisson’s ratio is positive.

Although the Poisson’s ratio is defined on a uniaxial test, it can be used in any arbitrary deformation.

Back to the original question. The only non-zero principal stress is \(\sigma_{11}\), which can be calculated from Eqn. (117):

\[
\sigma_{11} = 2\mu \varepsilon_{11} + \lambda (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})
\]

or using the Poisson’s ratio:

\[
\varepsilon_{22} = \varepsilon_{33} = -\nu \varepsilon_{11}
\]

so that

\[
\sigma_{11} = 2\mu \varepsilon_{11} + \lambda (\varepsilon_{11} - \nu \varepsilon_{11} - \nu \varepsilon_{11}) = (2\mu + \lambda (1 - 2\nu))\varepsilon_{11}
\]

From Eqn. (125):

\[
\nu = \frac{\lambda}{2(\lambda + \mu)}
\]

so that

\[
\sigma_{11} = (2\mu + \lambda (1 - \frac{2\lambda}{2(\lambda + \mu)}))\varepsilon_{11} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \varepsilon_{11} = E \varepsilon_{11}
\]

So if one is conducting a uniaxial test, and the only quantities of interest are axial strain and stress, then only a single material property is required - the Young’s modulus.

This example is, of course, the main justification for using the pair of the Young’s modulus and the Poisson’s ratio as the two linear elastic isotropic material constants.

For the pure shear example refer to ex. prob. 61. Note that the strain tensor can be expressed as:

\[
\varepsilon = \begin{pmatrix}
0 & \varepsilon_{12} & 0 \\
0 & 0 & \varepsilon_{12} \\
\text{sym} & 0 & 0
\end{pmatrix}
\]

from Eqn. (117) one immediately obtains that the only non-zero stress is:

\[
\sigma_{12} = 2\mu \varepsilon_{12}
\]

or, in shear modulus notation:

\[
\sigma_{12} = 2G \varepsilon_{12}
\]

from which the motivation for the shear modulus is obvious. This material property describes resistance to shear.
S74. From ex. prob. 73, in uniaxial stress state

\[ \varepsilon_{22} = \varepsilon_{33} = -\nu \varepsilon_{11} \]

So that

\[ I^\varepsilon = \text{tr} \varepsilon = \varepsilon_{11}(1 - 2\nu) \]

If \( \nu = 0.5 \) then \( I^\varepsilon = 0 \), and the deformation is incompressible. Because in this case incompressible deformation is solely due to a special material property, materials with \( \nu = 0.5 \) are called incompressible materials. Rubbers have \( \nu = 0.5 \). The assumption of incompressibility simplifies analysis of such materials.

S75. Can use any of Eqns. (109), (112), (114) or (115). Choose any normal stress, and express it via strain, e.g. for \( \sigma_{33} \):

\[ \sigma_{33} = \cdots + 2C_{3312}\varepsilon_{12} + 2C_{3323}\varepsilon_{23} + 2C_{3331}\varepsilon_{31} \]

which clearly shows that the normal stresses depend on shear strains.

Now choose any shear stress, e.g. \( \sigma_{12} \):

\[ \sigma_{12} = C_{1211}\varepsilon_{11} + C_{1222}\varepsilon_{22} + C_{1233}\varepsilon_{33} + \cdots \]

which clearly shows that the shear stresses depend on normal strains.

If one used \( \mathbf{S} \) instead, then normal strains are shown to depend on shear stresses and vice versa.

S76. As in the previous example can use any of Eqns. (109), (112), (114) or (115). Now imagine the strain tensor in principal coordinates. Try expressing any shear stress component via strain, e.g. for \( \sigma_{13} \):

\[ \sigma_{13} = C_{1311}\varepsilon_{11} + C_{1322}\varepsilon_{22} + C_{1333}\varepsilon_{33} \]

so in general there will be non-zero shear stresses corresponding to principal strains.

If one uses \( \mathbf{S} \) instead, then non-zero shear strains are shown to exist, corresponding to principal stress state.

S77. Can use either Eqns. (116) or (117). Choose any normal stress, e.g. \( \sigma_{22} \):

\[ \sigma_{22} = 2\mu \varepsilon_{22} + \lambda (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \]

so normal stresses depend only on normal strains. Choosing any shear stress, e.g. \( \sigma_{12} \):

\[ \sigma_{12} = 2\mu \varepsilon_{12} \]

As before, if one uses \( \mathbf{S} \) instead, then stresses and strains change places, but the conclusion is the same.

S78. The answer is immediately clear from Eqns. (116) or (117), or from ex. prob. 77. Assume the strain tensor in principal coordinates. Since shear strains are zero and shear stresses depend only on shear strains, then shear stresses are zero too. Therefore we conclude that the stress tensor is also in the principal CS. Hence the principal directions of strain and stress coincide.

S79. The aim is to express the strain tensor via the stress tensor. First, use (117) to calculate \( \text{tr} \sigma = \sigma_{kk} \).

\[
\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = (2\mu \varepsilon_{11} + \lambda \varepsilon_{kk}) + (2\mu \varepsilon_{22} + \lambda \varepsilon_{kk}) + (2\mu \varepsilon_{33} + \lambda \varepsilon_{kk}) \\
= 2\mu \varepsilon_{kk} + 3\lambda \varepsilon_{kk} = 2\mu \varepsilon_{kk} + 3\lambda \varepsilon_{kk} = (2\mu + 3\lambda)\varepsilon_{kk}
\]

or using the bulk modulus notation, \( K \):

\[ \sigma_{kk} = 3K \varepsilon_{kk} \]

This expression is notable for it relates the first invariant of strain, \( I^\varepsilon = \text{tr} \varepsilon = \varepsilon_{kk} \), to the first invariant of stress, \( I^\sigma = \text{tr} \sigma = \sigma_{kk} \). We have said earlier that the meaning of \( I^\varepsilon \) is volumetric strain. Now we can give the meaning to \( I^\sigma \). Its negative is called pressure:

\[ p = -\sigma_{kk} \]
so that positive pressure represents negative stress, which is consistent with our sign convention. The physical meaning of \( p \) is precisely hydrostatic pressure, except \( p \) can also be tensile. \( p \) can also be described as equi-triaxial stress state. The meaning of \( I^e = 3K^e \) is that pressure produces only a change of volume, never a change of shape. This is why \( K \) is called the *bulk modulus*.

With this Eqn. (117) can be rewritten as

\[
\sigma_{ij} = 2\mu \varepsilon_{ij} + \frac{\lambda}{3K} \sigma_{kk} \delta_{ij}
\]

Now we can express \( \varepsilon_{ij} \) as:

\[
\varepsilon_{ij} = \frac{1}{2\mu} \left( \sigma_{ij} - \frac{\lambda}{3K} \sigma_{kk} \delta_{ij} \right)
\]

or moving to the \( E \) and \( \nu \) pair of constants, and using Eqns. (126) and (127), we obtain:

\[
\varepsilon_{ij} = \frac{1 + \nu}{E} \left( \delta_{ip} \delta_{jq} - \frac{\nu}{1 + \nu} \delta_{ij} \delta_{pq} \right) \sigma_{pq}
\]

So that \( S \) can be written as:

\[
S_{ijpq} = \frac{1}{E} \left( (1 + \nu) \delta_{ip} \delta_{jq} - \nu \delta_{ij} \delta_{pq} \right)
\]

**S80.** Imagine a silicon sealant gun. Assume that the material of the pressure vessel is much stiffer (higher Young’s modulus) than silicon, which is a reasonable assumption. Then the walls of the pressure vessel can be considered rigid. Hence no strain in the direction normal to the axis of the vessel is possible. Consider point A somewhere sufficiently far from the nozzle. The only strain is compressive axial strain:

![Diagram of pressure vessel with point A](https://via.placeholder.com/150)

\[
\varepsilon = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \varepsilon_3 \\
\text{sym} & \varepsilon_3
\end{pmatrix}
\]

The stress tensor is found from Eqn. (117):

\[
\sigma = \begin{pmatrix}
\lambda \varepsilon_3 & 0 & 0 \\
0 & \lambda \varepsilon_3 & 0 \\
\text{sym} & (2\mu + \lambda) \varepsilon_3
\end{pmatrix}
\]

**S81.** A uniaxial strain state example is shown in ex. prob. 80. In a uniaxial stress state the stress tensor is:

\[
\sigma = \begin{pmatrix}
\sigma_{11} & 0 & 0 \\
0 & 0 & 0 \\
\text{sym} & 0
\end{pmatrix}
\]
if the stress is tensile.

From Eqn. (151) in ex. prob. 79 the corresponding strain tensor will be:

$$
\varepsilon = \frac{1}{E} \begin{pmatrix}
\sigma_{11} & 0 & 0 \\
0 & -\nu \sigma_{11} & 0 \\
-\nu \sigma_{11} & 0 & -\nu \sigma_{11}
\end{pmatrix}
$$

Observe that the factors connecting stress and strain components differ between uniaxial stress and uniaxial strain states. In particular, for directions of non-zero stress and strain we have for uniaxial strain:

$$
\sigma_{33} = (2\mu + \lambda) \varepsilon_{33}
$$

and for uniaxial stress:

$$
\sigma_{11} = E \varepsilon_{11}
$$

However, $E \neq 2\mu + \lambda$, see Eqn. (125).

Similarly, for strains in directions where there are no stresses, and for stresses in directions where there are no strains one has for uniaxial strain:

$$
\sigma_{11} = \lambda \varepsilon_{33}
$$

and for uniaxial stress:

$$
\sigma_{11} = -\frac{E}{\nu} \varepsilon_{22}
$$

Not only $\lambda \neq E/\nu$, see Eqn. (126), but here even the signs are opposite.

**S82.** First express $\sigma$ via $u$:

$$
\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij}
$$

where

$$
\varepsilon_{ij} = \frac{1}{2} (u_{1,i} + u_{j,i})
$$

so that

$$
\sigma_{ij} = \mu (u_{1,i} + u_{j,j}) + \lambda u_{k,k} \delta_{ij}
$$

and

$$
\sigma_{ij} = \mu (u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij}
$$

or using the properties of dummy indices and of the Kronecker delta tensor:

$$
\sigma_{ij} = \mu (u_{i,j} + u_{k,k}) + \lambda u_{k,k} \delta_{ij}
$$

or in tensor notation:

$$
\nabla \cdot \sigma = \mu \nabla^2 u + (\mu + \lambda) \nabla \cdot u
$$

Finally the equilibrium equations will be written as:

$$
\mu u_{1,j} + (\mu + \lambda) u_{k,k} = -b_i + \rho \ddot{x}_i
$$

or in tensor notation:

$$
\mu \nabla^2 u + (\mu + \lambda) \nabla \cdot u = -b + \rho \dot{x}
$$

**S83.** Remember that the order of differentiation is not important. Consider first components of the strain tensor in a particular plane, e.g. 12:

$$
\varepsilon_{11} = u_{1,1}
$$

$$
\varepsilon_{12} = \frac{1}{2} (u_{1,2} + u_{2,1})
$$
Differentiate $\epsilon_{11}$ over $x_2$ twice:

$$\epsilon_{11,22} = u_{1,122}$$

Now note that the same term is obtained if $\epsilon_{12}$ is differentiated over $x_1$ and $x_2$:

$$\epsilon_{12,12} = \frac{1}{2}(u_{1,212} + u_{2,112})$$

Now note that the second term in the last expression is obtained if $\epsilon_{22}$ is differentiated twice over $x_1$:

$$\epsilon_{22,11} = u_{2,211}$$

Comparing the last 3 expressions one sees that:

$$\epsilon_{11,22} + \epsilon_{22,11} = 2\epsilon_{12,12} \quad (152)$$

A further two expression of this kind are obtained if one considers strains in 23 and 31 planes. Alternatively one can just do a cyclic permutation of indices:

$$\epsilon_{22,33} + \epsilon_{33,22} = 2\epsilon_{23,23} \quad (153)$$

$$\epsilon_{33,11} + \epsilon_{11,33} = 2\epsilon_{31,31} \quad (154)$$

Now consider differentiating $\epsilon_{11}$ over $x_2$ and $x_3$:

$$\epsilon_{11,23} = u_{1,123} = u_{1,312}$$

The same term can be obtained if $\epsilon_{13}$ is differentiated over $x_1$ and $x_2$:

$$\epsilon_{13,12} = \frac{1}{2}(u_{1,312} + u_{3,112}) = \frac{1}{2}(u_{1,312} + u_{3,211})$$

Again looking at the last term, one sees that it can be obtained if $\epsilon_{32}$ is differentiated over $x_1$ twice:

$$\epsilon_{32,11} = \frac{1}{2}(u_{3,211} + u_{2,311}) = \frac{1}{2}(u_{3,211} + u_{2,131})$$

Again looking at the last term we can see that it is obtained if $\epsilon_{21}$ is differentiated over $x_3$ and $x_1$:

$$\epsilon_{21,31} = \frac{1}{2}(u_{2,131} + u_{1,231})$$

Note that the very last term is the same as the starting term, in $\epsilon_{11,23}$. Thus the chain is complete and we can link all four second derivatives of these strain components:

$$\epsilon_{13,12} - \epsilon_{32,11} = \frac{1}{2}(u_{1,312} - u_{2,131})$$

and

$$\epsilon_{11,23} - \epsilon_{21,31} = \frac{1}{2}(u_{1,312} - u_{2,131})$$

so that

$$\epsilon_{13,12} - \epsilon_{32,11} = \epsilon_{11,23} - \epsilon_{21,31}$$

or

$$\epsilon_{13,12} + \epsilon_{12,31} = \epsilon_{23,11} + \epsilon_{11,23} \quad (155)$$

As before, another 2 expressions can be obtained by cyclic permutation of all indices:

$$\epsilon_{21,23} + \epsilon_{23,21} = \epsilon_{31,22} + \epsilon_{22,31} \quad (156)$$

$$\epsilon_{32,31} + \epsilon_{31,32} = \epsilon_{12,33} + \epsilon_{33,12} \quad (157)$$

We leave it without proof that no further independent expressions linking derivatives of the components of the strain tensor exist.

If Eqns. (152)-(154) are rewritten as:

$$\epsilon_{11,22} + \epsilon_{22,11} = \epsilon_{12,12} + \epsilon_{12,12}$$
\[ \varepsilon_{22,33} + \varepsilon_{33,22} = \varepsilon_{23,23} + \varepsilon_{23,33} \]
\[ \varepsilon_{33,11} + \varepsilon_{11,33} = \varepsilon_{31,31} + \varepsilon_{31,11} \]

then a pattern emerges. The 6 equations can be summarised as:

\[ \varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik} \]

Note that of these 81 equations 75 are redundant, i.e. repeated or trivial, as for \( i = j = k = l = 1 \).

**S84.** The key to the solution is to remember that the stress tensor is linked to the stress vector, and vectors are additive.

From Eqn. (64):

\[ \mathbf{\sigma} \cdot \mathbf{n} = t^a \]

Consider an element of the boundary of the body under analysis, \( \Gamma_t \), where traction is applied:

When traction \( t^{(1)} \) is applied, the stress tensor at that point is:

\[ \mathbf{\sigma}^{(1)} \cdot \mathbf{n} = t^{(1)} \]

When traction \( t^{(2)} \) is applied, the stress tensor at that point is:

\[ \mathbf{\sigma}^{(2)} \cdot \mathbf{n} = t^{(2)} \]

Traction vectors can be added to produce the total traction

\[ t^a = t^{(1)} + t^{(2)} = \mathbf{\sigma}^{(1)} \cdot \mathbf{n} + \mathbf{\sigma}^{(2)} \cdot \mathbf{n} = (\mathbf{\sigma}^{(1)} + \mathbf{\sigma}^{(2)}) \cdot \mathbf{n} \]

On the other hand the stress tensor corresponding to \( t^a \) is

\[ \mathbf{\sigma} \cdot \mathbf{n} = t^a \]

From the last two equations one immediately obtains:

\[ \mathbf{\sigma} = \mathbf{\sigma}^{(1)} + \mathbf{\sigma}^{(2)} \]

which means that the stress states are additive.

An alternative proof is based on the equilibrium equations, Eqn. (67):

\[ \nabla \cdot \mathbf{\sigma} = -\mathbf{b} + \rho \ddot{\mathbf{x}} \]

The key property of this PDE is that it is linear, meaning that if \( \mathbf{\sigma}^{(1)} \) and \( \mathbf{\sigma}^{(2)} \) are two solutions to these PDEs, then \( \mathbf{\sigma} = \mathbf{\sigma}^{(1)} + \mathbf{\sigma}^{(2)} \) is a solution too. This is a consequence of the fact that the differential operator is linear:

\[ \nabla \cdot \mathbf{\sigma} = \nabla \cdot \mathbf{\sigma}^{(1)} + \nabla \cdot \mathbf{\sigma}^{(2)} \]

**S85.** The proof proceeds similar to ex. prob. 84. Need to remember that displacements are vectors and therefore additive.

Consider a complex displacement path - displacement \( \mathbf{u}^{(1)} \), followed by displacement \( \mathbf{u}^{(2)} \):
Each displacement vector will have a corresponding strain tensor:

\[ \varepsilon^{(1)} = \frac{1}{2}(\nabla u^{(1)} + (\nabla u^{(1)})^T) \]

\[ \varepsilon^{(2)} = \frac{1}{2}(\nabla u^{(2)} + (\nabla u^{(2)})^T) \]

The total displacement will give rise to its own strain tensor:

\[ \varepsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) \]

The gradient (and derivative in general) is a distributive operator, meaning

\[ \nabla(a + b) = \nabla a + \nabla b \]

or in index notation

\[ a_{ij} + b_{ij} = \frac{\partial a}{\partial x_i} + \frac{\partial b}{\partial x_i} = \frac{\partial(a + b)}{\partial x_i} = (a + b)_{ij} \]

hence by adding strains from displacements \( u^{(1)} \) and \( u^{(2)} \) one obtains:

\[ \varepsilon^{(1)} + \varepsilon^{(2)} = \frac{1}{2}(\nabla u^{(1)} + \nabla u^{(2)} + (\nabla u^{(1)})^T + (\nabla u^{(2)})^T) = \frac{1}{2}(\nabla(u^{(1)} + u^{(2)}) + (\nabla u^{(1)})^T + (\nabla u^{(2)})^T) \]

The transposition is a distributive operator too, meaning

\[ (A + B)^T = A^T + B^T \]

so that

\[ \varepsilon^{(1)} + \varepsilon^{(2)} = \frac{1}{2}(\nabla(u^{(1)} + u^{(2)}) + (\nabla(u^{(1)} + u^{(2)}))^T) = \frac{1}{2}(\nabla u + (\nabla u)^T) = \varepsilon \]

Hence strain states are additive.

Note that the proof in index notation is easier in this case:

\[ \varepsilon_{ij}^{(1)} = \frac{1}{2}(u_{i,j}^{(1)} + u_{j,i}^{(1)}) \]

\[ \varepsilon_{ij}^{(2)} = \frac{1}{2}(u_{i,j}^{(2)} + u_{j,i}^{(2)}) \]

So that

\[ \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)} = \frac{1}{2}(u_{i,j}^{(1)} + u_{i,j}^{(1)} + u_{i,j}^{(2)} + u_{j,i}^{(2)}) = \frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ij} \]

**S86.** The stress tensor in principal directions is:

\[
\sigma = \begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
\text{sym} & 0 & 0
\end{pmatrix}
\]

The principal direction with zero stress is 3. A rotation tensor with a single rotation about 3 is:

\[
R = \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The rotated stress tensor is:

\[ \sigma' = R\sigma R^T \]
In the following we are not interested in the exact values, only whether they are zero or not. We use Greek symbols for non-zero values.

\[
R\sigma = \begin{pmatrix}
\alpha & \beta & 0 \\
\gamma & \delta & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

and

\[
R\sigma R^T = \begin{pmatrix}
\eta & \zeta & 0 \\
\kappa & \omega & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

So we proved that for any rotation in 12 plane \(\sigma_{13} = \sigma_{23} = \sigma_{33} = 0\).

**S87.** This problem has a point symmetry. Cut a sphere along a diameter into two halves, and use force equilibrium along the axis normal to the cut, 1:

We assume that the membrane thickness, \(t \ll r\), where \(r\) is the radius of the sphere, so that the membrane thickness can be neglected when the sphere diameter is calculated. \(S_1\) is the half sphere surface. \(S_2\) is the surface of the cross section circle.

The horizontal force acting on the membrane due to pressure is

\[
\int_{S_1} p\mathbf{v} \cdot \mathbf{n} dS
\]

where \(\mathbf{n}\) is the outward normal vector to \(S_1\). \(\mathbf{v}\) is the unit vector pointing in \(-1\) direction. Note that this integral can be interpreted as flux of vector \(p\mathbf{v}\) through \(S_1\).

Flux of this vector through \(S_2\) is

\[
\int_{S_2} p\mathbf{v} \cdot \mathbf{n} dS = -p\pi r^2
\]

because \(S_2\) is just an area of a circle, and \(p = \text{const}\).

The total flux though \(S_1 \cup S_2\) is

\[
\int_{S_1 \cup S_2} p\mathbf{v} \cdot \mathbf{n} dS = \int_{S_1} p\mathbf{v} \cdot \mathbf{n} dS - p\pi r^2
\]

the left hand side can be transformed into a volume integral by Green’s theorem:

\[
\int_V p\mathbf{v}_i dV = \int_{S_1} p\mathbf{v} \cdot \mathbf{n} dS - p\pi r^2 = 0
\]

where \(V\) is the volume bounded by \(S_1 \cup S_2\). The volume integral is zero because \(\mathbf{v} = \text{const}\). Hence

\[
\int_{S_1} p\mathbf{v} \cdot \mathbf{n} dS = p\pi r^2
\]

The other horizontal force acting on the membrane is due to stress in the membrane:
\[
p\pi r^2 = 2\pi rt\sigma_{11}
\]
So that
\[
\sigma_{11} = \frac{pr}{2t}
\]
From symmetry, \(\sigma_{22} = \sigma_{11}\) and all shear stresses are zero. So the stress state is equibiaxial tension. At any point in the membrane any two orthogonal directions in the plane of the membrane are principal. The third principal direction is through thickness.

The stress tensor is:
\[
\sigma = \frac{pr}{2t}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \text{sym}
\end{pmatrix}
\]
Strain through thickness is calculated from Eqn. (151) in ex. prob. 79:
\[
\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) = -\frac{\nu pr}{Et}
\]
The reduction in thickness is:
\[
t\varepsilon_{33} = -\frac{\nu pr}{E}
\]
Note that this solution is very crude, because it uses linear elasticity and small strain, linear, formulation. Neither is true in the case of a rubber balloon. Hence this solution might give unrealistic prediction. For example, if \(E = 10\text{MPa}\), \(\nu = 0.5\), \(r = 20\text{mm}\), \(t = 0.2\text{mm}\), and \(p = 0.2\text{MPa}\), i.e. about twice the atmospheric pressure, then \(\varepsilon_{33} = -1\) and the reduction in thickness is \(-0.2\text{mm}\), meaning the thickness after the deformation is zero.

**S88.** In general, plane stress state produces non-zero third principal strain. From Eqn. (151) in ex. prob. 79:
\[
\varepsilon_3 = -\frac{\nu}{E}(\sigma_1 + \sigma_2)
\]
For this to be zero one must have \(\sigma_1 = -\sigma_2\).

Plane strain state gives rise to non-zero third principal stress:
\[
\sigma_3 = \lambda(\varepsilon_1 + \varepsilon_2)
\]
This is zero if \(\varepsilon_1 = -\varepsilon_2\).

Both conditions can be rephrased as
\[
\text{tr}\sigma = \text{tr}\varepsilon = I^\sigma = I^\varepsilon = 0
\]
meaning that the spherical parts of both the strain and the stress tensors are zero. Yet in other words, - there must be neither volumetric strain, no pressure stress. Such stress/strain state is pure shear. The stress and strain tensors in principal coordinates will be:
\[
\sigma = \begin{pmatrix}
\sigma_1 & 0 \\
\text{sym} & -\sigma_1
\end{pmatrix}
; \quad \varepsilon = \begin{pmatrix}
\varepsilon_1 & 0 \\
\text{sym} & -\varepsilon_1
\end{pmatrix}
\]
or in the coordinates of maximum shear value:
Therefore, pure shear is a special case of both plane strain and plane stress conditions.

S89. The bracket has a right angle and is clamped at the wall. The lengths of each part of the bracket are \( l \). The view in 13 plane is:

Beam AB is along axis 3, and beam BO is along axis 1. It is clear that AB is loaded in bending, while BO is under a combined bending and torsion. Therefore AB seems easier to analyse so we start from it.

The bending moment in AB looks like this:

From Eqn.(29) in Sec. 4.4

\[
\frac{d^2w}{dx_3^2} = \frac{M}{EI}
\]

or

\[
w'' = \frac{P(l - x_3)}{EI}
\]

By integrating it twice one obtains:

\[
w' = \frac{P}{EI} \int (l - x_3)dx_3 + C_1 = \frac{P}{EI} (lx_3 - \frac{x_3^2}{2}) + C_1
\]

\[
w = \frac{P}{EI} \int \int (l - x_3)dx_3dx_3 + C_1x_3 + C_2 = \frac{P}{EI} (l \frac{x_3^2}{2} - \frac{x_3^3}{6}) + C_1x_3 + C_2
\]

The integration constants are found from the BC:

\[w(x_3 = 0) = 0 \Rightarrow C_2 = 0\]

Note that this BC is correct only when one considers AB in isolation. Clearly \( w_B \neq 0 \) for the complete problem, because it arises from the deflection of BO. However, \( w_B \) can be added as a rigid body motion to all points in AB, after it is calculated from the analysis of BO.

We now see that there is no easy second BC to fit \( C_1 \). Indeed \( w_B' \) is not known. The BC at point B is a compatibility condition joining the two beams together. This means this problem is of a statically indeterminate type, meaning that extra information must be used to calculate the BC for AB. Clearly, torsion of BO determines \( w_B' \). So let’s now move to the analysis of BO.

Let’s start with bending.
The solution for \( w \) is identical to that for AB. One only needs to substitute \( x_1 \) for \( x_3 \):

\[
w_{BO} = P \frac{E}{I} \left( l \frac{x_1^2}{2} - \frac{x_3^3}{6} \right) + C_1 x_1 + C_2
\]

However, now we have two easy BC:

\[
w_{BO}(x_1 = 0) = 0 \implies C_2 = 0
\]
\[
w_{BO}'(x_1 = 0) = 0 \implies C_1 = 0
\]

so that

\[
w_{BO} = P \frac{x_1^2}{6EI} (3l - x_1)
\]

The torsion of BO is due to force \( P \) applied off axis. The torque is \( T = Pl \). From Eqn. (138) in Sec. 6.1.4 the twist angle is:

\[
\theta = \int \frac{T}{GJ} dx_1 = \int \frac{Pl}{GJ} dx_1
\]

so that

\[
\theta_{\text{max}} = \frac{Pl^2}{GJ}
\]

In the following, let’s assume an axisymmetric cross section, for which \( I_{11} = I_{22} \), and hence (see ex. prob. 9):

\[
J = 2l
\]

Also, from Eqn. (126):

\[
G = \mu = \frac{E}{2(1+v)}
\]

so that \( \theta_{\text{max}} \) can be rewritten as:

\[
\theta_{\text{max}} = \frac{Pl^2}{EI} (1 + \nu)
\]

The extra BC is:

\[
w_{AB}'(x_3 = 0) = \theta_{\text{max}}
\]

so

\[
C_{1}^{AB} = \frac{Pl^2}{EI} (1 + \nu)
\]

Finally, \( w \) on AB, \( w_{AB} \) is:

\[
w_{AB} = \frac{P}{EI} \left( \frac{x_3^2}{2} - \frac{x_3^3}{6} + l^2 x_3(1 + \nu) \right)
\]

We can now calculate displacement in the complete system. For this one needs to add \( w_{BO}^{\text{max}} \) to \( w_{AB} \):

\[
w_{BO}^{\text{max}} = \frac{Pl^3}{3EI}
\]

so that \( w_{AB} \) is:
The maximum displacement in the system is at $A$:

$$w_{AB}^{\text{max}} = w_{AB}(x_3 = l) = \frac{Pl^3}{EI} \left(\frac{5}{3} + \nu\right)$$

The deformed shape of the bracket can now be drawn. The initial (undeformed) shape is shown with dashed lines. The new shape is shown with solid lines. The dotted line is $w_{BO}^{\text{max}}$, which is added to $w_{AB}$ as a rigid body motion. The magnitudes are shown in units of $Pl^3/EI$.

Let’s now turn to the analysis of stress, first in $AB$. From Eqn. (28) in Sec. 4.4 the maximum axial stress in the cross section is:

$$\sigma_{33} = \frac{M_{x2}^{\text{max}}}{I}$$

and the maximum moment is at $x_3 = 0$. Let’s also use $R_o$ for $x_2^{\text{max}}$, since we have already assumed an axisymmetric cross section.

$$\sigma_{33}^{\text{max}} = \frac{P|R_o|}{I}$$

We have studied pure bending already, it is enough to say that the stress (and the strain) states are uniaxial, with the maximum principal stress

$$\sigma_1 = \sigma_{33}^{\text{max}} = \frac{P|R_o|}{I}$$

Stress in BO is a superposition (see ex. probs. 84, 85) of stress tensors due to bending and torsion:

$$\sigma_{BO} = \sigma_{BO}^{\text{bend}} + \sigma_{BO}^{\text{torsion}}$$

It is clear that the sum is maximised at $x_2 = \pm R_o$, where the torsion stress acts in 13 plane. In all other points either $\sigma_{11}$ or the shear stresses, or both, are smaller.
hence:

\[ \sigma_{BO} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{13} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{sym}} + \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & 0 \\ \sigma_{11} & 0 & 0 \end{pmatrix}_{\text{sym}} = \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} \\ 0 & 0 & 0 \\ \sigma_{11} & 0 & \sigma_{13} \end{pmatrix}_{\text{sym}} \]

Along \( x_1 \) the stress due to torsion is constant, and that due to bending is highest at \( x_1 = 0 \), where it is the same as for AB:

\[ \sigma_{11}^{\text{max}} = \frac{PIR_0}{I} \]

From Eqns. (135) and (137) in Sec. 6.1.4 the maximum torsion shear stress is

\[ \sigma_{13}^{\text{max}} = G\theta' R_0 = \frac{GPIR_0}{GJ} = \frac{PIR_0}{2I} \]

Finally

\[ \sigma_{BO} = \frac{PIR_0}{I} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \text{sym} & 0 & 0 \end{pmatrix} \]

Note that we solved this problem so easily only because it could be reduced to a 2D superposition problem.

The stress tensor can be shown on the elementary cube of material, in the units of \( PIR_0/I \), as:

To find the principal values and directions I use again Lapack DSYEV routine:

Original tensor

\[
\begin{array}{ccc}
1.00000000000E+00 & 0.00000000000E+00 & 5.00000000000E-01 \\
0.00000000000E+00 & 0.00000000000E+00 & 0.00000000000E+00 \\
5.00000000000E-01 & 0.00000000000E+00 & 0.00000000000E+00 \\
\end{array}
\]

The DSYEV eigenvalues in increasing order

\[-2.07106781187E-01 \quad 0.00000000000E+00 \quad 1.20710678119E+00 \]

The DSYEV orthonormal eigenvectors (columns)

\[
\begin{array}{ccc}
3.82683432365E-01 & 0.00000000000E+00 & 9.23879532511E-01 \\
-0.00000000000E+00 & -1.00000000000E+00 & 0.00000000000E+00 \\
-9.23879532511E-01 & 0.00000000000E+00 & 3.82683432365E-01 \\
\end{array}
\]

The angles (deg)
The stress tensor in the original and the principal CS is best shown in 2D representation:

\[ \sigma_1 = 1.2, \sigma_3 = -0.2, \text{ both in units of } \frac{P l R_o}{I}, \text{ and the rotation is } 22.5^\circ, \text{ as expected. So the maximum principal stress in the bracket is at point O:} \]

\[ \sigma_O^1 = 1.2 \frac{P l R_o}{I} \]

Alternatively one can use the Mohr’s diagram, as the stress state is two-dimensional:

Note that some very complex stress/strain state will exist around point B, where all our assumptions of bending and torsion will be violated. Typically portions of designs which are too hard to analyse are made deliberately stronger than the rest.

Finally, let’s get a feel for the values of displacement and stress for typical engineering values. Consider a ring cross section with the outer radius \( R_o = 20\text{mm} \) and wall thickness \( t = 2\text{mm} \), a typical steel with \( E = 200\text{GPa} \) and \( \nu = 0.33 \), the bracket size \( l = 500\text{mm} \), loaded by \( P = 1\text{kN} \). Then

\[ w_{\text{max}} = w_A = 29\text{mm} \]

\[ \sigma_{\text{max}}^1 = \sigma_O^1 = 278\text{MPa} \]

**S90.** To show that \( S_1 \) and \( S_2 \) form a vector, one has to show that they change with CT as components of a vector.

In matrix form coordinates of the centroid can be written as:

\[
\begin{bmatrix}
S_1 \\
S_2
\end{bmatrix} = \frac{1}{A} \begin{bmatrix}
i_2 \\
i_1
\end{bmatrix}
\]

\[
S_1' = \frac{1}{A} i_2' = \frac{1}{A} \int R_{1j} x_j dA = \frac{1}{A} R_{1j} \int x_j dA = \frac{1}{A} \left(R_{11} i_2' + R_{12} i_1' \right)
\]

Similarly
\[ S_2' = \frac{1}{A} (R_{21}i_2 + R_{22}i_1) \]

Hence

\[
\begin{align*}
S_1' &= \frac{1}{A} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} i_2 \\ i_1 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} S_1 \\
S_2' &= \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} S_2
\end{align*}
\]

Hence \( S = S_j = (S_1, S_2) \) is a vector.

**S91.** The easiest logic follows from ex. prob. 90. For any CS with origin at centroid \( S = 0 \). Hence \( i_1 = i_2 = 0 \).

Note that the first moments of area strictly do not form a vector. This is because:

\[
i_1' = \int R_{2j} x_j dA = R_{2j} \int x_j dA \neq R_{1j} \int x_j dA
\]

The transformation is similar to that of vector components, but not the same.

**S92.** Slight complication arises because

\[
I_{ij} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}
\]

is not a tensor. Indeed

\[
I_{11}' = \int x_1' x_2' dA = \int R_{2j} x_j dA = R_{2j} \int x_j dA = R_{1j} \int x_j dA
\]

which is not how R2T components should transform. However:

\[
Y_{ij} = \begin{pmatrix} I_{22} & I_{12} \\ I_{21} & I_{11} \end{pmatrix}
\]

is indeed R2T, which is easily proved:

\[
Y_{mn}' = \begin{pmatrix} I_{22}' & I_{12}' \\ I_{21}' & I_{11}' \end{pmatrix}
\]

For example, for \( Y_{22}' \) one obtains:

\[
Y_{22}' = I_{11}' = \int x_2' x_2' dA = \int R_{2j} x_j dA = R_{2j} \int x_j dA
\]

\[
= R_{21} R_{21} \int x_1 x_1 dA + R_{21} R_{22} \int x_1 x_2 dA + R_{22} R_{21} \int x_2 x_1 dA + R_{22} R_{22} \int x_2 x_2 dA
\]

\[
= R_{21} R_{21} I_{22} + R_{21} R_{22} I_{12} + R_{22} R_{21} I_{21} + R_{22} R_{22} I_{11}
\]

\[
= R_{21} R_{21} Y_{11} + R_{21} R_{22} Y_{12} + R_{22} R_{21} Y_{21} + R_{22} R_{22} Y_{22} = R_{21} R_{22} Y_{ij}
\]

which is how components of R2T should transform. The same can be shown for the other components.

\( Y = Y_{ij} \) is symmetrical, because \( \int x_1 x_2 dA = \int x_2 x_1 dA \).

We call \( Y \) the second moment of area tensor.

One can argue that the historical definitions of the second moments of area are wrong, and it would be more logical to define \( I_{11} \) as \( \int x_1 x_1 dA \), etc. In that case \( I_{ij} \) would be R2T, and none of this confusion would arise.

**S93.** Construct the second moment of area tensor \( Y \) using the expressions from ex. prob. 12:
\[
Y = \begin{pmatrix}
\frac{HW^3}{36} & -\frac{W^2H^2}{72} \\
\frac{W^2H^2}{72} & \frac{WH^3}{36}
\end{pmatrix} = \begin{pmatrix}
HW & 2W^2 & -HW \\
-HW & 2H^2
\end{pmatrix}
\]

Note that \(W\) is the side length along 1 and \(H\) is the side length along 2.

One can construct the quadratic characteristic equation:

\[
(2W^2 - \lambda)(2H^2 - \lambda) - W^2H^2 = 0
\]

or

\[
\lambda^2 - 2\lambda(H^2 + W^2) + 3W^2H^2 = 0
\]

So the principal values are:

\[
\lambda_{1,2} = H^2 + W^2 \pm \left( (H^2 - W^2)^2 + W^2H^2 \right)^{\frac{1}{2}}
\]

Expressions for the principal directions become quite long, so to make the example more instructive, let’s assume

\[H = 2W\]

which means the other two angles in the triangle are 30° and 60°. Then \(Y\) will simplify to:

\[
Y = \frac{W^4}{18} \begin{pmatrix}
1 & -1 \\
-1 & 4
\end{pmatrix}
\]

I solve the eigenvalue/eigenvector problem numerically with LAPACK library, routine DSYEV:

Original tensor

| 0.00000000000E+00 | 0.00000000000E+00 | 0.00000000000E+00 |
| 0.00000000000E+00 | 1.00000000000E+00 | -1.00000000000E+00 |
| 0.00000000000E+00 | -1.00000000000E+00 | 4.00000000000E+00 |

The DSYEV eigenvalues in increasing order

| 0.00000000000E+00 | 6.97224362268E-01 | 4.30277563773E+00 |

The DSYEV orthonormal eigenvectors (columns)

| 1.00000000000E+00 | 0.00000000000E+00 | 0.00000000000E+00 |
| 0.00000000000E+00 | -9.57092026489E-01 | -2.89784186888E-01 |
| 0.00000000000E+00 | -2.89784186888E-01 | 9.57092026489E-01 |

The angles (deg)

| 9.00000000000E+01 | 9.00000000000E+01 | 9.00000000000E+01 |
| 9.00000000000E+01 | 1.63154966237E+02 | 1.06845033763E+02 |
| 9.00000000000E+01 | 1.68450337630E+01 | 1.68450337630E+01 |

The principal directions are shown below.
Note that the result is not obvious. In particular, the principal axis is not \( \parallel \) to the hypotenuse.

The second moment of area tensor in principal directions is:

\[
\mathbf{Y} = \frac{W^4}{18} \begin{pmatrix} 4.303 & 0 \\ 0 & 0.697 \end{pmatrix}
\]

The practical importance of this example is that it allows finding the beam orientation for maximum resistance to bending.

**S94.** Use the fact that the second moments form \( \mathbb{R}^2 \). If symmetry axes are used as CS then \( I_{11} = I_{22} \) and \( I_{12} = 0 \). Hence \( I_1 = I_2 = I_{11} \), i.e. both principal values are identical. Therefore any axis passing through centroid is a principal axis.

Prepared with groff, Xfig and gnuplot.